ONLINE APPENDIX:

TRADEOFFS AND COMPARISON COMPLEXITY

F Additional Proofs

F.1 Characterization Results

Proof of Theorem 2.

The proof of necessity is routine. Theorem 1 covers sufficiency for the $n \geq 3$ case. We now show sufficiency in the case where $n = 2$; assume that M1–M6 hold. Note that Claim 1 in the proof of Theorem 1 continues to hold in this case; that is, that for any $z \in \mathbb{R}^n$ satisfying $\sum_k z_k \geq 0$, $\rho(z, 0) = \rho(d^+(z)e_1 - d^-(z)e_2, 0)$. To see this, note that if $z_1 \geq 0$, $z_2 \geq 0$, the desired equality follows from Dominance; if not then either i) $z_1 > 0$, $z_2 < 0$ or ii) $z_1 < 0$ and $z_2 > 0$. In case i), the equality is immediate since $z = d^+e_1 + d^-e_2$, which in conjunction with Exchangeability, implies the desired equality for case ii). Following the steps in Claims 2 and 3 in the proof of Theorem 1 completes the proof of sufficiency. Note that the argument for uniqueness in Theorem 1 holds for $n = 2$, and so uniqueness holds as well.

□

Proof of Theorem 3.

The proof of necessity of M1, M4–M5, and M7 are routine. To see that M3 (Moderate Transitivity) is necessary, consider x, y, z with $\rho(x, y) \geq 1/2$ and $\rho(y, z) \geq 1/2$. If $d_{L1}(x, y), d_{L1}(y, z)$, $d_{L1}(x, z) > 0$, then the restriction of ρ to $\{x, y, z\}$ belongs to the moderate utility class studied in He and Natenzon (2023a) and so by Theorem 1 of this paper we can conclude that this restriction satisfies Moderate Transitivity. There are four additional cases to consider. Case 1: suppose $d_{L1}(x, y) = 0$. We then have $\rho(x, z) = \rho(y, z)$ and $\rho(x, y) = 1/2$, so either $\rho(x, z) > \min\{\rho(x, y), \rho(y, z)\}$ or $\rho(x, z) = \rho(y, z) = \rho(x, z)$. Case 2: $d_{L1}(y, z) = 0$. We then have $\rho(x, z) = \rho(x, y)$ and $\rho(y, z) = 1/2$, and so again either $\rho(x, z) > \min\{\rho(x, y), \rho(y, z)\}$ or $\rho(x, z) = \rho(y, z) = \rho(x, z)$. Case 3: $d_{L1}(x, z) = 0$. Here we have $\rho(x, z) = 1/2$, and $\rho(x, y) = \rho(z, y) \geq 1/2$ and $\rho(y, z) \geq 1/2$, which implies $\rho(y, z) = \rho(x, y) = 1/2$; we therefore have $\rho(x, y) = \rho(y, z) = \rho(x, z)$. Finally, consider $d_{L1}(x, y) = d_{L1}(x, z) = d_{L1}(y, z) = 0$; here we have $\rho(x, y) = \rho(y, z) = \rho(x, z)$, and so Moderate Transitivity holds in all cases.

To see that M8 (Tradeoff Congruence) is necessary, take $(x,y),(y,z) \in \mathcal{D}$ congruent such that $\rho(x,y),\rho(y,z) \geq 1/2$. Note that if $d_{L1}(x,z)=0$, then $\rho(x,z)=1/2$ and since ρ satisfies Moderate Transitivity we have $\rho(x,y)=\rho(y,z)=1/2$ and we are done. Now consider the case where $d_{L1}(x,z) \neq 0$. Note that

$$\begin{aligned}\rho(x,z) &= G\left(\frac{\sum_k (u_k(x_k)-u_k(z_k))}{\sum_k |u_k(x_k)-u_k(z_k)|}\right)\\ &= G\left(\frac{\sum_k (u_k(x_k)-u_k(y_k)+u_k(y_k)-u_k(z_k))}{\sum_k |u_k(x_k)-u_k(y_k)+u_k(y_k)-u_k(z_k)|}\right)\\ &= G\left(\frac{U(x)-U(y)+U(y)-U(z)}{d_{L1}(x,y)+d_{L1}(y,z)}\right)\end{aligned}$$

Where the final equality holds because congruence implies that $u_k(x_k)-u_k(y_k)$ and $u_k(y_k)-u_k(z_k)$ must either be both positive or negative. This implies that if either $d_{L1}(x,y)=0$ or $d_{L1}(y,z)=0$, we are done. Now consider the case where $d_{L1}(x,y), d_{L1}(y,z) > 0$, and suppose $\rho(y,z) \leq \rho(x,y)$; this implies $\frac{U(y)-U(z)}{d_{L1}(y,z)} \leq \frac{U(x)-U(y)}{d_{L1}(x,y)}$. The above implies

$$\begin{aligned}\rho(x,z) &= G\left(\frac{\frac{U(x)-U(y)}{d_{L1}(y,z)} + \frac{U(y)-U(z)}{d_{L1}(y,z)}}{\frac{d_{L1}(x,y)}{d_{L1}(y,z)}+1}\right)\\ &\leq G\left(\frac{\frac{U(x)-U(y)}{d_{L1}(y,z)} + \frac{U(x)-U(y)}{d_{L1}(x,y)}}{\frac{d_{L1}(x,y)}{d_{L1}(y,z)}+1}\right)\\ &= \rho(x,y)\end{aligned}$$

and so $\rho(x,z) \leq \max\{\rho(x,y),\rho(y,z)\}$ when $\rho(y,z) \leq \rho(x,y)$. The argument for the case where $\rho(y,z) \geq \rho(x,y)$ is analogous.

Now we show sufficency. Let $\succeq$ be the stochastic preference relation induced by ρ. $\succeq$ satisfies coordinate independence and inherits continuity from ρ, and since we have at least 3 non-null attributes, we invoke Debreu (1983) to conclude that $\succeq$ has an additively separable representation: there exists $u_i : X_i \to \mathbb{R}$, continuous, such that

$$x \succeq y \iff \sum_k u_k(x_k) \geq \sum_k u_k(y_k)$$

Since all attributes are non-null and the X_k are connected, each $u_k(X_k)$ is a non-trivial interval of $\mathbb{R}$. Since the representation is unique up to cardinal transformations, we can without loss assume that for each $k \in I$, $u_k(X_k)$ contains 0, and furthermore, since $u_k(X_k)$ is a non-trivial interval, that $u_k(X_k)$ contains a non-trivial open interval around 0. For all $k \in I$, let $\overline{u}_k = \sup u_k(X_k)$ and $\underline{u}_k = \inf u_k(X_k)$, taken with respect to the extended real line, and let $\Delta_k = \overline{u}_k - \underline{u}_k$. For all $x \in X$, define $\tilde{x} = (u_1(x_1), ..., u_k(x_k)) \in \mathbb{R}^n$. Begin by noting the following result.

Lemma 8. *For $x, y \in X$ with $\tilde{x} = \tilde{y}$: $\rho(x,z) = \rho(y,z)$ for all $z \in X$.*

Proof. Fix such an x, y, and take any $z \in X$. Note that $x \sim y$ by hypothesis. First consider the case where $x \sim y \succeq z$. Since (x, y) and (y,z) are congruent, and likewise (y, x) and (x,z) are congruent, Tradeoff Congruence implies

$$\begin{aligned} \rho(x,z) &\leq \max\{\rho(y,z), \rho(x,y)\} = \rho(y,z) \\ \rho(y,z) &\leq \max\{\rho(x,z), \rho(y,x)\} = \rho(x,z) \end{aligned}$$

and so $\rho(y,z) = \rho(x,z)$. Analogously, consider the case where $z \succeq x \sim y$. Since (z, x) and (x, y) are congruent and likewise (z, y) and (y, x) are congruent, we have

$$\begin{aligned} \rho(z,x) &\leq \max\{\rho(z,y), \rho(y,x)\} = \rho(z,y) \\ \rho(z,y) &\leq \max\{\rho(z,x), \rho(x,y)\} = \rho(z,x) \end{aligned}$$

and so $\rho(z,x) = \rho(z,y) \implies \rho(x,z) = \rho(y,z)$. $\square$

Let $\tilde{X} = \{\tilde{x} \in \mathbb{R}^n : x \in X\}$. Let $\tilde{\mathcal{D}} = \{(a,b) \in \tilde{X} : a \neq b\}$ and define $\phi : \tilde{\mathcal{D}} \to \mathcal{D}$ satisfying $\phi(a,b) \in \{(x,y) \in \mathcal{D} : \tilde{x} = a, \tilde{y} = b\}$, and define $\tilde{\rho} : \tilde{\mathcal{D}} \to [0,1]$ by $\tilde{\rho}(a,b) = \rho(\phi(a,b))$. Lemma 8 implies that $\tilde{\rho}$ is a binary choice rule on $\tilde{\mathcal{D}}$ and does not depend on the selection made by ϕ: in particular, we have $\tilde{\rho}(\tilde{x},\tilde{y}) = \rho(x,y)$ for all $(x,y) \in \mathcal{D}$. This in turn implies that $\tilde{\rho}$ inherits our axioms M1,M3–M5, M7–M8. Note that if there exists a strictly increasing, continuous function G such that

$$\tilde{\rho}(a,b) = G\left(\frac{\sum_k (a_k - b_k)}{\sum_k |a_k - b_k|}\right)$$

for all $(a,b) \in \tilde{\mathcal{D}}$, we are done, as this implies that for any $(x,y) \in \mathcal{D}$ such that $\tilde{x} \neq \tilde{y} \iff \sum_k |u_k(x_k) - u_k(y_k)| > 0$,

$$\rho(x,y) = \tilde{\rho}(\tilde{x},\tilde{y}) = G\left(\frac{\sum_k (u_k(x_k) - u_k(y_k))}{\sum_k |u_k(x_k) - u_k(y_k)|}\right)$$

and furthermore for $(x, y) \in \mathcal{D}$ such that $\tilde{x} = \tilde{y}$, we have $x \sim y \implies \rho(x, y) = 1/2$, and so ρ has an additively separable L_1-complexity representation.

In what follows, we will work with $\tilde{\rho}$ defined on $\tilde{X}$ and suppress the $\sim$ in our notation. Say that ρ defined on this domain is

- *Translation invariant* if for all $x, x', y, y' \in X, z \in \mathbb{R}^n$ such that $x' = x + z$, $y' = y + z$, $\rho(x', y') = \rho(x, y)$.
- *Scale invariant* if for all $x, x', y, y' \in X$ such that $x' = cx, y' = cy$ for $c > 0$, $\rho(x', y') = \rho(x, y)$.
- *Translation invariant** if for all $x, x', y, y' \in X, z \in \mathbb{R}^n$ such that $x' = x + z$, $y' = y + z$, and additionally $x_k = y_k$ for some $k \in I$, $\rho(x', y') = \rho(x, y)$.
- *Scale invariant** if for all $x, x', y, y' \in X$ such that $x' = cx, y' = cy$ for $c > 0$, and additionally $x_k = y_k$ for some $k \in I$, $\rho(x', y') = \rho(x, y)$.
- *Translation invariant*† if for all $x, x', y, y' \in X, z \in \mathbb{R}^n$ such that $x' = x + z$, $y' = y + z$, and additionally $x_k = y_k$ for some $k \in I$ such that $|x_i - y_i| < \Delta_k$ for all $i \in I$, $\rho(x', y') = \rho(x, y)$.
- *Scale invariant*† if for all $x, x', y, y' \in X$ such that $x' = \lambda x, y' = \lambda y$ for $\lambda \in (0, 1)$, and additionally $x_k = y_k$ for some $k \in I$ such that $|x_i - y_i| < \Delta_k$ for all $i \in I$, $\rho(x', y') = \rho(x, y)$.

First, note that Separability and Simplification imply translation invariance†.

Lemma 9. *Suppose ρ satisfies Separability and Simplification. Then ρ satisfies translation invariance†.*

Proof. Begin by noting that for $x', y', x, y \in X$, $z \in \mathbb{R}^n$ with $x' = x + z$, $y' = y + z$, and $x_k = y_k$ for some $k \in I$ such that $|x_i - y_i| < \Delta_k$ for all $i \in I$: for any $E \subseteq I$, $x + \sum_{j \in E} z_{\{j\}}$ and $y + \sum_{j \in E} z_{\{j\}}$ will be in our domain, with $\left(x + \sum_{j \in E} z_{\{j\}}\right)_k = \left(y + \sum_{j \in E} z_{\{j\}}\right)_k$ and with $\left|\left(x + \sum_{j \in E} z_{\{j\}}\right)_i - \left(y + \sum_{j \in E} z_{\{j\}}\right)_i\right| < \Delta_k$ for all i. Since we can translate x and y by each component $z_{\{j\}}$ attribute-by-attribute, it suffices to show that for any $x, y \in X$ with $x_k = y_k$ where $|x_i - y_i| < \Delta_k$ for all $i \in I$, $z \in \mathbb{R}^n$, $j \in I$ such that $x + z_{\{j\}}$ and $y + z_{\{j\}}$ belong to our domain, $\rho(x + z_{\{j\}}, y + z_{\{j\}}) = \rho(x, y)$. Fix such an $x, y \in X$, $z \in \mathbb{R}^n$, $k, j \in I$.

Note that if $j = k$, Separability gives us the desired result. Now suppose $j \neq k$. Suppose that $x_j \geq y_j$ (the argument for $x_j < y_j$ is analogous). For any $i \in I$, $a \in (\underline{u}_i, \overline{u}_i)$, $w \in X$, let $a_{\{i\}}w \in X$ denote the option equal to a for attribute $k = i$ and equal to w_k for all other attributes. Since by hypothesis $|x_i - y_i| < \Delta_k$ for all i, there exists some $b \in (\underline{u}_k, \overline{u}_k)$ such that $|x_i - y_i| < \overline{u}_k - b$ for all i. By Separability, we have $\rho(b_{\{k\}}x, b_{\{k\}}y) = \rho(x, y)$. Now consider $x' \in \mathbb{R}^n$ satisfying

$$x'_i = \begin{cases} y_j & i = j \\ b + (x_j - y_j) & i = k \\ x_i & \text{otherwise} \end{cases}$$

By construction, $b + (x_j - y_j) < \overline{u}_k$, and so $x' \in X$. Applying Simplification twice, we have $\rho(x', b_{\{k\}}y) = \rho(b_{\{k\}}x, b_{\{k\}}y)$. Since $x'_j = (b_{\{k\}}y)_j$ by construction, Separability in turn implies that $\rho(x' + z_{\{j\}}, b_{\{k\}}y + z_{\{j\}}) = \rho(x', b_{\{k\}}y)$. Again applying Simplification twice, we have $\rho(b_{\{k\}}x + z_{\{j\}}, b_{\{k\}}y + z_{\{j\}}) = \rho(x' + z_{\{j\}}, b_{\{k\}}y + z_{\{j\}})$. A final application of Separability yields $\rho(x + z_{\{j\}}, y + z_{\{j\}}) = \rho(b_{\{k\}}x + z_{\{j\}}, b_{\{k\}}y + z_{\{j\}})$, and the chain of equalities yields $\rho(x + z_{\{j\}}, y + z_{\{j\}}) = \rho(x, y)$ as desired. □

The next result says that scale invariance* is implied by translation invariance† and our other axioms.

Lemma 10. *Suppose ρ satisfies translation invariance†, Continuity, Moderate Transitivity, and Tradeoff Congruence. Then ρ satisfies scale invariance*.*

Proof. First, show that invariance† holds for half-mixtures and then extend the result to arbitrary mixtures using continuity. In particular, we want to show that for $x, y \in X$ with $x_k = y_k$ for some k such that $|x_i - y_i| < \Delta_k$ for all i, $\rho(x, y) = \rho(\frac{1}{2}x, \frac{1}{2}y)$. Without loss, suppose that $x \succeq y$. By translation invariance†, we have $\rho(x, \frac{1}{2}x + \frac{1}{2}y) = \rho(\frac{1}{2}x + \frac{1}{2}y, y) = \rho(\frac{1}{2}x, \frac{1}{2}y)$. Since $(x, \frac{1}{2}x + \frac{1}{2}y)$ and $(\frac{1}{2}x + \frac{1}{2}y, y)$ are congruent and $x \succeq \frac{1}{2}x + \frac{1}{2}y \succeq y$, by Tradeoff Congruence and Moderate Transitivity, we have $\rho(x, y) = \rho(x, \frac{1}{2}x + \frac{1}{2}y) = \rho(\frac{1}{2}x, \frac{1}{2}y)$ as desired.

We now show that for any $x, y \in X$ with $x_k = y_k$ and $|x_i - y_i| < \Delta_k$ for all $i \in I$, for any $n \in \mathbb{N}$, $\rho(x, y) = \rho(\alpha x, \alpha y)$ for all $\alpha \in \{\frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{2^n}{2^n}\}$. Note that if $x \sim y$, then the result holds by definition of $\succeq$ and we are done. Now suppose that $x \not\sim y$, and assume without loss that $x \succ y$. Proceed inductively; given what we have shown above, the statement is true for $n = 1$. Now suppose the statement is true for some n; we wish to show that for any $m \in \{1, \ldots, 2^{n+1}\}$, $\rho(\frac{m}{2^{n+1}}x, \frac{m}{2^{n+1}}y) = \rho(x, y)$. Note that for any $m \leq 2^n$ we have

$\rho(\frac{m}{2^{n+1}}x, \frac{m}{2^{n+1}}y) = \rho(\frac{m}{2^n}x, \frac{m}{2^n}y) = \rho(x, y)$ using our result on half-mixtures and by inductive hypothesis.

Now consider $m \in \{2^n + 1, ..., 2^{n+1}\}$. Note that by translation invariance† and by inductive hypothesis, we have $\rho(\frac{m}{2^{n+1}}x, \frac{1}{2}y + \frac{m-2^n}{2^{n+1}}x) = \rho(\frac{1}{2}x, \frac{1}{2}y) = \rho(x, y)$. Also, by translation invariance† and inductive hypothesis, we have $\rho(\frac{1}{2}y + \frac{m-2^n}{2^{n+1}}x, \frac{m}{2^{n+1}}y) = \rho(\frac{m-2^n}{2^{n+1}}x, \frac{m-2^n}{2^{n+1}}y) = \rho(x, y)$. These two equalities and Moderate Transitivity imply that $\rho(\frac{m}{2^{n+1}}x, \frac{m}{2^{n+1}}y) \geq \rho(x, y)$.

Toward a contradiction, suppose $\rho(\frac{m}{2^{n+1}}x, \frac{m}{2^{n+1}}y) > \rho(x, y)$. Translation invariance† then implies $\rho(x, \frac{2^{n+1}-m}{2^{n+1}}x + \frac{m}{2^{n+1}}y) > \rho(x, y)$. By translation invariance† and the result shown above, we also have $\rho(\frac{2^{n+1}-m}{2^{n+1}}x + \frac{m}{2^{n+1}}y, y) = \rho(\frac{2^{n+1}-m}{2^{n+1}}x, \frac{2^{n+1}-m}{2^{n+1}}y) = \rho(x, y)$. But since Moderate Transitivity implies that $\rho(x, y) > \rho(\frac{2^{n+1}-m}{2^{n+1}}x + \frac{m}{2^{n+1}}y, y)$, we have a contradiction. This proves the statement for $n+1$, and so by induction the statement holds for any n. By taking limits and by Continuity of ρ, we can then conclude that scale invariance† holds.

Now we show that scale invariance* holds. Fix any $x, y \in X$ where $x_k = y_k$ for some k. Without loss, assume $x \succeq y$. First, show that $\rho(x, y) = \rho(\lambda x, \lambda y)$ for any $\lambda \in (0, 1)$. Note that there exists some $N \in \mathbb{N}$ such that $\frac{1}{N}|x_i - y_i| < \Delta_k$ for all i. For $n \in \{0, 1, ..., N\}$, define $w^n \in X$ by $w^n = \frac{n}{N}x + \frac{N-n}{N}y$. Now consider the sequence of comparisons (w^N, w^{N-1}), $(w^{N-1}, w^{N-2}), ..., (w^1, w^0)$. Since $w^n - w^{n-1} = \frac{1}{N}(x - y)$ for all n, we have $w^n \succeq w^{n-1}$ for all n, and additionally $|w_i^n - w_i^{n-1}| < \Delta_k$ for all i, and so translation invariance† implies that for all n, $\rho(w^n, w^{n-1}) = \rho(w^n - (\frac{N-n}{N}y + \frac{n-1}{N}x), w^{n-1} - (\frac{N-n}{N}y + \frac{n-1}{N}x)) = \rho(\frac{1}{N}x, \frac{1}{N}y)$. Sequential applications of Moderate Transitivity and Tradeoff Congruence yield, respectively

$$\rho(x, y) \geq \min\{\rho(w^N, w^{N-1}), \rho(w^{N-1}, w^{N-2}), ..., \rho(w^1, w^0)\}$$
$$\rho(x, y) \leq \max\{\rho(w^N, w^{N-1}), \rho(w^{N-1}, w^{N-2}), ..., \rho(w^1, w^0)\}$$

and so we have $\rho(x, y) = \rho(\frac{1}{N}x, \frac{1}{N}y)$. An analogous argument, taking the sequence of comparisons $(\lambda w^N, \lambda w^{N-1})$, $(\lambda w^{N-1}, \lambda w^{N-2}), ..., (\lambda w^1, \lambda w^0)$, yields $\rho(\lambda x, \lambda y) = \rho(\lambda \frac{1}{N}x, \lambda \frac{1}{N}y)$. By scale invariance†, noting again that $\frac{1}{N}|x_i - y_i| < \Delta_k$ for all i, we have $\rho(\lambda \frac{1}{N}x, \lambda \frac{1}{N}y) = \rho(\frac{1}{N}x, \frac{1}{N}y)$ and so $\rho(x, y) = \rho(\lambda x, \lambda y)$ as desired.

We have therefore shown that for any $x, y \in X$ with $x_k = y_k$ for some k, $\lambda \in (0, 1)$, $\rho(x, y) = \rho(\lambda x, \lambda y)$. Finally, fix some $c > 0$ and $x, y \in X$ with $x_k = y_k$ for some k and $cx, cy \in X$; we wish to show that $\rho(x, y) = \rho(cx, cy)$. If $c \leq 1$, we are done by the result established above. If instead $c > 1$, the above result implies that $\rho(cx, cy) = \rho(\frac{1}{c}cx, \frac{1}{c}cy) = \rho(x, y)$. $\square$

Scale invariance* allows us to strengthen translation invariance† to translation invariance*.

Lemma 11. *Suppose ρ satisfies translation invariance† and scale invariance*. Then ρ satisfies translation invariance*.*

Proof. Take $x, y \in X$ with $x_k = y_k$ for some k, and $z \in \mathbb{R}^n$ such that $x + z, y + z \in X$. There exists some $\lambda \in (0, 1)$ such that $\lambda|x_i - y_i| < \Delta_k$ for all i; fix such a λ. We then have

$$\begin{aligned} \rho(x, y) &= \rho(\lambda x, \lambda y) \\ &= \rho(\lambda(x + z), \lambda(y + z)) \\ &= \rho(x + z, y + z) \end{aligned}$$

where the first and third equalities use scale invariance* and the second equality uses translation invariance†. □

We now show that scale invariance*, translation invariance*, and Tradeoff Congruence imply translation invariance.

Lemma 12. *Suppose ρ satisfies translation invariance*, scale invariance*, Simplification, Tradeoff Congruence, and Moderate Transitivity. Then ρ satisfies translation invariance.*

Proof. Take any $x, y \in X$, $w \in \mathbb{R}^n$ such that $x + w, y + w \in X$. We want to show that $\rho(x + w, y + w) = \rho(x, y)$. Without loss, assume that $x \succeq y$. Note that if $x \geq y$, we are done by Dominance, so consider the case where $x \not\geq y$. Let $z = x - y \in \mathbb{R}^n$. If $z_k = 0$ for some k, then by translation invariance* we are done, so consider the case where $z_k \neq 0$ for all k. It must then be the case that there exist distinct indices $i, j \in I$ such that $\text{sgn}(z_i) = \text{sgn}(z_j) \neq 0$. Define $z^i, z^j \in \mathbb{R}^n$ such that

$$z^i_k = \begin{cases} z_i + z_j & k = i \\ 0 & k = j \\ z_k & \text{otherwise} \end{cases} \qquad z^j_k = \begin{cases} 0 & k = i \\ z_i + z_j & k = j \\ z_k & \text{otherwise} \end{cases}$$

Letting $\lambda = \frac{z^i}{z^i + z_j} \in (0, 1)$, note that by construction $z = \lambda z^i + (1 - \lambda) z^j$. Now fix any $v \in X$ such that $z + v, v \in X$; note that $z + v \in X \implies (1 - \lambda) z^j + v \in X$. Since each $u_k(X_k)$ contains a non-trivial open interval around 0, there exists $\gamma \in (0, 1)$ such that $\gamma z^i, \gamma z^j \in X$. We then have

$$\begin{aligned}
\rho(z+v,(1-\lambda)z^j+v) &= \rho(\gamma(z+v),\gamma((1-\lambda)z^j+v)) \\
&= \rho(\gamma\lambda z^i,0) \\
&= \rho(\gamma z^i,0) \\
&= \rho(\gamma z^j,0) \\
&= \rho(\gamma(1-\lambda)z^j,0) \\
&= \rho(\gamma((1-\lambda)z^j+v),\gamma v) \\
&= \rho((1-\lambda)z^j+v,v)
\end{aligned}$$

Where the first three equalities follow from scale invariance* and translation invariance*, noting that by construction, $(1-\lambda)z^j_j = z_j$, the fourth equality follows from two applications of Simplification, and the final three equalities follow from translation invariance* and scale invariance*, noting that $z^j_i = 0$.

By construction, $(z+v,(1-\lambda)z^j+v)$ and $((1-\lambda)z^j+v,v)$ are congruent, since $[z+v]-[(1-\lambda)z^j+v] = \lambda z^i$ and $[(1-\lambda)z^j+v]-v = (1-\lambda)z^j$, and since for all k, either $z^j_k, z^i_k \geq 0$ or $z^j_k, z^i_k \leq 0$. Furthermore, since $\sum_k z^i_k = \sum_k z^j_k = \sum_k z_k \geq 0$, we have $z+v \succeq (1-\lambda)z^j+v$ and $(1-\lambda)z^j+v \succeq v$. We then have

$$\begin{aligned}
\rho(z+v,v) &= \rho(z+v,(1-\lambda)z^j+v) \\
&= \rho(\gamma z^i,0)
\end{aligned}$$

Where the first equality follows from Tradeoff Congruence and Moderate Transitivity, and the second equality follows from the chain of equalities above. Since this equality holds for all v such that $z+v, v \in X$, substituting $v = y$ and $v = y+w$ yields $\rho(x,y) = \rho(x+w,y+w)$ as desired. □

Lemma 13. *Suppose ρ satisfies translation invariance, Continuity, Moderate Transitivity, and Tradeoff Congruence. Then ρ satisfies scale invariance.*

Proof. Fix any $x,y \in X$, and without loss suppose $x \succeq y$. Note that by translation invariance, we have $\rho(x,\frac{1}{2}x+\frac{1}{2}y) = \rho(\frac{1}{2}x+\frac{1}{2}y,y) = \rho(\frac{1}{2}x,\frac{1}{2}y)$. Since $(x,\frac{1}{2}x+\frac{1}{2}y)$ and $(\frac{1}{2}x+\frac{1}{2}y,y)$ are congruent and $x \succeq \frac{1}{2}x+\frac{1}{2}y \succeq y$, by Tradeoff Congruence and Moderate Transitivity, we have $\rho(x,y) = \rho(x,\frac{1}{2}x+\frac{1}{2}y) = \rho(\frac{1}{2}x,\frac{1}{2}y)$.

The proof for extending the result on half-mixtures to arbitrary mixtures and then to arbitrary rescaling follows an analogous argument as in the proof for Lemma 10, invoking translation invariance whenever translation invariance† is invoked in that proof. $\square$

Using Lemmas 9–13, we conclude that ρ satisfies scale and translation invariance. Linearly extend ρ to $\mathbb{R}^n$ as follows. Define $\overline{\mathcal{D}} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$, and define $\overline{\rho} : \overline{\mathcal{D}} \to [0, 1]$ such that for any $(x, y) \in \mathcal{D}$, $\overline{\rho}(x, y) = \rho(x, y)$, and for any $(x, y) \in \overline{\mathcal{D}} \setminus \mathcal{D}$, $\rho(x, y) = \rho(\lambda x, \lambda y)$ for some $\lambda \in (0, 1)$ such that $\lambda x, \lambda y \in X$. Since X contains an open a ball around the origin, this extension is well-defined. Furthermore, since ρ satisfies scale and translation invariance, so does $\overline{\rho}$, and so $\overline{\rho}$ satisfies M2 (Linearity). Noting that for any finite collection of options $A \subseteq \mathbb{R}^n$, there exists $\lambda \in (0, 1)$ such that $\lambda x \in X$ for all $x \in A$, by scale invariance of $\overline{\rho}$ it is straightforward to show that $\overline{\rho}$ is a binary choice rule and satisfies M1, M3–M5. Theorem 1 then implies that there exists G continuous, strictly increasing, such that for all $(x, y) \in \overline{\mathcal{D}}$,

$$\overline{\rho}(x, y) = G\left(\frac{\sum_k (x_k - y_k)}{\sum_k |x_k - y_k|}\right)$$

which in turn implies that for all $(x, y) \in \mathcal{D}$,

$$\rho(x, y) = \overline{\rho}(x, y) = G\left(\frac{\sum_k (x_k - y_k)}{\sum_k |x_k - y_k|}\right)$$

which yields the desired representation.

Finally, we show uniqueness. Suppose that ρ has additively separable L_1 complexity representations $((u_i)_{i=1}^n, G)$ and $((u'_i)_{i=1}^n, G')$. Let $\succeq$ denote the stochastic order on X induced by ρ. Since G and G' are strictly increasing and symmetric around 0, we have for all $x, y \in X$

$$x \succeq y \iff \sum_k u_k(x_k) \geq \sum_k u_k(y_k) \iff \sum_k u'_k(x_k) \geq \sum_k u'_k(y_k)$$

and U, U' both represent $\succeq$, where $U(x) = \sum_k u_k(x_k)$ and $U'(x) = \sum_k u'_k(x_k)$. Debreu (1983) then implies that there exists $C > 0$, $b_k \in \mathbb{R}$ such that $u'_k = C u_k + b_k$ for all k. This implies that for all $x, y \in X$,

$$G\left(\frac{\sum_k (u_k(x_k) - u_k(y_k))}{\sum_k |u_k(x_k) - u_k(y_k)|}\right) = G'\left(\frac{\sum_k (u_k(x_k) - u_k(y_k))}{\sum_k |u_k(x_k) - u_k(y_k)|}\right)$$

By assumption, there exist two non-null indices; without loss, we assume indices 1 and 2 are non-null. Since u_1, u_2 are continuous and X_1 and X_2 are connected, $u_1(X_1)$ and $u_2(X_2)$

are intervals in $\mathbb{R}^n$. Since we have shown that the u_k are unique up to affine transformations, we can without loss assume that for all $\mu \in [0,1]$, there exist $x_1^\mu \in X_1$ and $x_2^\mu \in X_2$ such that $u_1(x_1^\mu) = u_2(x_2^\mu) = \mu$.

Fix some $\overline{x} \in X$. For any $\alpha, \gamma \in [0,1]$, note that for $x, y \in X$ with

$$x_k = \begin{cases} x_1^\alpha & k=1 \\ x_2^0 & k=2 \\ \overline{x}_k & \text{otherwise} \end{cases} \qquad y_k = \begin{cases} x_1^0 & k=1 \\ x_2^\gamma & k=2 \\ \overline{x}_k & \text{otherwise} \end{cases}$$

we have

$$\rho(x,y) = G\left(\frac{\alpha-\gamma}{\alpha+\gamma}\right) = G'\left(\frac{\alpha-\gamma}{\alpha+\gamma}\right)$$

Since for any $r \in [-1,1]$ there exists $\alpha, \gamma \in [0,1]$ such that $\frac{\alpha-\gamma}{\alpha+\gamma} = r$, we must have $G' = G$.

Proof of Theorem 4.

Necessity of the axioms is immediate from the definition; we now show sufficiency.

Let $\succeq$ denote the stochastic order on X induced by ρ. By Moderate Transitivity, $\succeq$ is transitive. Since ρ satisfies Continuity and Independence, $\succeq$ satisfies the vNM axioms and so there exists a utility function $u : \mathbb{R} \to \mathbb{R}$ such that $U(x) = \sum_w u(w) f_x(w)$ represents $\succeq$; Dominance implies that u is strictly increasing.

Fix any four distinct prizes $w_a, w_b, w_c, w_d \in \mathbb{R}$ such that $u(w_a) > u(w_b) > u(w_c) > u(w_d)$. Consider any two lotteries $x, y \in X$. Enumerate $S_x \cup S_y \cup \{w_a, w_b, w_c, w_d\}$ by $w_1, w_2, \ldots, w_{n+1}$, where $w_1 < w_2 < \ldots < w_{n+1}$, and let $K = \{1, \ldots, n, n+1\}$. Let $X(K)$ denote the set of finite-state lotteries with support on $\{w_1, w_2, \ldots, w_{n+1}\}$. With some abuse of notation, we let a, b, c, d denote the indices in K corresponding to prizes w_a, w_b, w_c, w_d. We have $u(w_1) < u(w_2) < \ldots < u(w_{n+1})$. With some abuse of notation, for any $z \in X(K)$, let $F_z(k) = \sum_{w \leq w_k} f_z(w)$ denote the value of the CDF of z at support point w_k, and let $u(k) = u(w_k)$.

Identify each lottery $z \in X(K)$ with its *utility-weighted* CDF vector $\tilde{z} \in \mathbb{R}^n$, where

$$\tilde{z}_k = -F_z(k)(u(k+1) - u(k))$$

for $k = 1, 2, ..., n$. Note that for any $x, y \in X(K)$,

$$\frac{\sum_k (\tilde{x}_k - \tilde{y}_k)}{\sum_k |\tilde{x}_k - \tilde{y}_k|} = \frac{U(x) - U(y)}{d_{CDF}(x, y)}$$

We now seek to extend the space of utility-weighted CDF vectors to $\mathbb{R}^n$ in order to apply Theorem 1. Let $\mu \in X(K)$ denote the lottery that is uniform over K; that is $F_\mu(k) = \frac{k}{n+1}$. Consider the set

$$V = \{a \in \mathbb{R}^n : a_k = \alpha(\tilde{x}_k - \tilde{\mu}_k) : x \in X(K), \alpha > 0)\}.$$

Lemma 14. $V = \mathbb{R}^n$.

Proof. By definition we have $V \subseteq \mathbb{R}^n$. To see that $\mathbb{R}^n \subseteq V$, take any $a \in \mathbb{R}^n$. We will show that $a \in V$. Define

$$\begin{aligned}
\beta &= \max_{k \in \{2,3,\ldots,n\}} (n+1)\left[a_k/(u(k+1) - u(k)) - a_{k-1}/(u(k) - u(k-1))\right] \\
\gamma &= (n+1)[a_1/(u(2) - u(1))] \\
\eta &= -(n+1)[a_n/(u(n+1) - u(n))]
\end{aligned}$$

and fix any $\alpha > \max\{\beta, \gamma, \eta, 0\}$. Define $H : K \to \mathbb{R}$ given by

$$H(k) = \begin{cases} F_\mu(k) - \dfrac{a_k/(u(k+1) - u(k))}{\alpha} & k < n+1 \\ 1 & k = n+1 \end{cases}$$

Since $\alpha > \beta$, we have $H(k+1) - H(k) \geq 0$ for all $k = 1, ..., n$, and since $\alpha > \eta$, we have $1 = H(n+1) - H(n) \geq 0$, and so H is increasing. Furthermore, since $\alpha > \gamma$, $H(1) \geq 0$, and so H is positive on its domain. Since $H(n+1) = 1$, H is the CDF of a lottery in $X(K)$, which we denote by x. Note that by construction, for all $k = 1, ..., n$ we have

$$\begin{aligned}
\alpha(\tilde{x}_k - \tilde{\mu}_k) &= \alpha\left(-F_\mu(k)(u(k+1) - u(k)) + \frac{a_k}{\alpha} + F_\mu(k)(u(k+1) - u(k))\right) \\
&= a_k
\end{aligned}$$

which implies that $a \in V$. $\square$

For any $a, b \in V$, let

$$L(a,b) = \{(x,y) \in X(K) \times X(K) : a = \alpha(\tilde{x} - \tilde{\mu}), b = \alpha(\tilde{y} - \tilde{\mu}),\ \alpha > 0\}.$$

Lemma 15. *Let $W \subseteq V$ finite. Then there exists $\alpha > 0$ such that for all $a \in W$, $a = \alpha(\tilde{x} - \tilde{\mu})$ for some $x \in X(K)$.*

Proof. Enumerate the elements of W by $\{a^1, a^2, ..., a^l\}$. For all $m = \{1, 2, ..., l\}$, there exists $\alpha^m > 0$, $z^m \in X(K)$ such that $a^m = \alpha^m(\tilde{z}^m - \tilde{\mu})$. Let $\alpha = \max_m \alpha^m$, and for all m, define $x^m \in X(K)$ satisfying $(\alpha^m/\alpha)z^m + (1 - \alpha^m/\alpha)\mu$, and notice that $a^m = \alpha(\tilde{x}^m - \tilde{\mu})$. $\square$

Define some $\phi : V \times V \to X(K) \times X(K)$ that takes an arbitrary selection from $L(a,b)$; Lemma 15 implies $L(a,b)$ is non-empty, ϕ is well-defined. For $\hat{\mathcal{D}} = \{(a,b) \in V \times V : a \neq b\}$, define $\hat{\rho} : \hat{\mathcal{D}} \to [0,1]$ by $\hat{\rho}(a,b) = \rho(\phi(a,b))$.

Lemma 16. *$\hat{\rho}$ is uniquely identified by ρ. That is, for any $a, b \in V$: for any $(x,y), (x',y') \in L(a,b)$,. $\rho(x,y) = \rho(x',y')$ and so $\hat{\rho}$ does not depend on the choice of ϕ. Also, $\hat{\rho}$ is a binary choice rule, that is, $\hat{\rho}(a,b) = 1 - \hat{\rho}(b,a)$.*

Proof. Fix some $a, b \in V$, and suppose $(x,y), (x',y') \in L(a,b)$. It suffices to show that $\rho(x,y) = \rho(x',y')$. Since $(x,y), (x',y') \in L(a,b)$, there exists $\alpha, \alpha' > 0$ such that

$$\begin{aligned} a &= \alpha(\tilde{x} - \tilde{\mu}) = \alpha'(\tilde{x}' - \tilde{\mu}) \\ b &= \alpha(\tilde{y} - \tilde{\mu}) = \alpha'(\tilde{y}' - \tilde{\mu}) \end{aligned}$$

Without loss, we can take $\alpha' > \alpha$. For $\lambda = \frac{\alpha}{\alpha'}$, the above inequalities directly imply that

$$\begin{aligned} x' &= \lambda x + (1-\lambda)\mu \\ y' &= \lambda y + (1-\lambda)\mu \end{aligned}$$

and so by Independence of ρ, $\rho(x,y) = \rho(x',y')$.

Finally to see that $\hat{\rho}$ is a binary choice rule, take any $a, b \in V$. By Lemma 15, there exists $\alpha > 0$, $x, y \in X(K)$ such that $a = \alpha(\tilde{x} - \tilde{\mu})$, $b = \alpha(\tilde{y} - \tilde{\mu})$; we have

$$\begin{aligned} \hat{\rho}(a,b) &= \rho(x,y) \\ &= 1 - \rho(y,x) \\ &= 1 - \hat{\rho}(b,a) \end{aligned}$$

as desired. $\square$

Lemma 17. *$\hat{\rho}(a,b) \geq 1/2 \iff \sum_k a_k \geq \sum_k b_k$., and $\hat{\rho}$ satisfies M1–M5.*

Proof. Fix any $a, b, c, a', b' \in V$. By Lemma 15, there exists $\alpha > 0$, $x, y, z, x', y' \in X(K)$ such that $a = \alpha(\tilde{x} - \tilde{\mu})$, $b = \alpha(\tilde{y} - \tilde{\mu})$, $c = \alpha(\tilde{z} - \tilde{\mu})$, $a' = \alpha(\tilde{x}' - \tilde{\mu})$, $b' = \alpha(\tilde{y}' - \tilde{\mu})$.

To show the first claim, note that $\hat{\rho}(a,b) \geq 1/2 \iff \rho(x,y) \geq 1/2 \iff U(x) \geq U(y) \iff \sum_k \tilde{x}_k \geq \sum_k \tilde{y}_k \iff \sum_k a_k \geq \sum_k b_k$.

To see that $\hat{\rho}$ satisfies Continuity, note that $\hat{\rho}$ inherits continuity from ρ. To see that $\hat{\rho}$ satisfies Linearity, take any $\lambda \in [0,1]$.Note that by construction, $\lambda a + (1-\lambda)c = \alpha(\lambda\tilde{x} + (1-\lambda)\tilde{z} - \tilde{\mu})$ and $\lambda b + (1-\lambda)c = \alpha(\lambda\tilde{y} + (1-\lambda)\tilde{z} - \tilde{\mu})$, and so

$$\begin{aligned}
\hat{\rho}(\lambda a + (1-\lambda)c, \lambda b + (1-\lambda)c) &= \rho(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)z) \\
&= \rho(x,y) \\
&= \hat{\rho}(a,b)
\end{aligned}$$

where the first and final equalities follow from Lemma 16, and the second equality follows from Independence of ρ.

To show that $\hat{\rho}$ satisfies Moderate Transitivity, suppose that $\hat{\rho}(a,b) \geq 1/2$, $\hat{\rho}(b,c) \geq 1/2$. This implies that $\rho(x,y) \geq 1/2$, $\rho(y,z) \geq 1/2$, and so Moderate Transitivity of ρ implies that $\rho(x,z) \geq \min\{\rho(x,y), \rho(y,z)\}$, which in turn implies that $\hat{\rho}(a,c) \geq \min\{\rho(a,b), \rho(b,c)\}$, and so $\hat{\rho}$ satisfies Moderate Transitivity.

To show that $\hat{\rho}$ satisfies Dominance, by Lemma 16, it suffices to show that if $a_k \geq b_k$ for all k, then $x \geq y$. To see this, suppose that $a_k \geq b_k$ for all k; this implies that $\tilde{x}_k \geq \tilde{y}_k$ for all k, which in turn implies that $F_x(k) \leq F_y(k)$ for all k, and so $x \geq y$.

Finally, to see that $\hat{\rho}$ satisfies Simplification, consider $a, b \in V$ with $\rho(a,b) \geq 1/2$ and a' satisfying $a'_i = b_i$, $a'_k \neq b_k$ for all $k \neq i, j$ for $i \neq j$, with $\rho(a', a) \geq 1/2$.

By Lemma 15, there exists $\alpha > 0$, $x, x', y \in X(K)$ such that $a = \alpha(\tilde{x} - \tilde{\mu})$, $a' = \alpha(\tilde{x}' - \tilde{\mu})$, $b = \alpha(\tilde{y} - \tilde{\mu})$, and Lemma 16 implies that $\rho(x,y) \geq 1/2$ and $\rho(x',x) \geq 1/2$. Define $\hat{x}, \hat{x}', \hat{y}$ by $\hat{x} = 1/2x + 1/2\mu$, $\hat{x}' = 1/2x' + 1/2\mu$, and $\hat{y} = 1/2y + 1/2\mu$. By construction that $S_{\hat{x}} = S_{\hat{x}'} = S_{\hat{y}} = \{w_1, ..., w_{n+1}\}$, and so in particular $S_{\hat{x}'} \subseteq S_{\hat{x}} \cup S_{\hat{y}}$. Independence implies that $\rho(\hat{x},\hat{y}) \geq 1/2$, $\rho(\hat{x}',\hat{x}) \geq 1/2$. Moreover, since $a_i' = b_i$, we have $F_{\hat{x}'}(w_i) = F_{\hat{y}}(w_i)$, and since $a_k' = a_k$ for all $k \neq j,i$, we have $F_{\hat{x}'}(w) = F_{\hat{x}}(w)$ for all $w \in S_{\hat{x}} \cup S_{\hat{y}}/\{w_i, w_j\}$. Since ρ satisfies Simplification, we have $\rho(\hat{x}',\hat{y}) \geq \rho(\hat{x},\hat{y})$. Independence then implies $\rho(x',y) \geq \rho(x,y)$, and so applying Lemma 16, we have $\hat{\rho}(a',b) \geq \hat{\rho}(a,b)$, and so $\hat{\rho}$ satisfies Simplification. □

Using Lemma 17, Theorem 1 then implies that there exists a continuous, strictly increasing $G : [-1,1] \to [0,1]$, symmetric around 0, such that for all $a, b \in \mathbb{R}^n$ we have

$$\hat{\rho}(a,b) = G\left(\frac{\sum_k (a_k - b_k)}{\sum_k |a_k - b_k|}\right)$$

Lemma 16 then implies that for any $x, y \in X(K)$, we have

$$\begin{aligned}\rho(x,y) &= \hat{\rho}(\tilde{x} - \tilde{\mu}, \tilde{y} - \tilde{\mu}) \\ &= G\left(\frac{\sum_k (\tilde{x}_k - \tilde{y}_k)}{\sum_k |\tilde{x}_k - \tilde{y}_k|}\right) \\ &= G\left(\frac{U(x) - U(y)}{d_{CDF}(x,y)}\right)\end{aligned}$$

Let $\mathcal{K} = \{K \subseteq S : |K| < \infty, \{w_a, w_b, w_c, w_d\} \subseteq K\}$. The above implies that for any $K \in \mathcal{K}$, there exists a continuous, strictly increasing $G_K : [-1,1] \to [0,1]$ such that for all $x, y \in X(K)$,

$$\rho(x,y) = G_K\left(\frac{U(x) - U(y)}{d_{CDF}(x,y)}\right)$$

All that remains is to show that for any $K, K' \in \mathcal{K}$, $G_K = G_{K'}$. To see this, fix any $K, K' \in \mathcal{K}$, and for $\alpha \geq 0$, $\gamma \geq 0$, consider $x, y \in X$ with

$$x = \begin{cases} w_b & \text{w.p. } 1 \end{cases} \qquad y = \begin{cases} w_c & \text{w.p. } \dfrac{\alpha/(u(w_b) - u(w_c))}{\alpha/(u(w_b) - u(w_c)) + \gamma/(u(w_a) - u(w_b))} \\ w_a & \text{w.p. } \dfrac{\gamma/(u(w_a) - u(w_b)}{\alpha/(u(w_b) - u(w_c)) + \gamma/(u(w_a) - u(w_b))} \end{cases}$$

Note that x, y belong to both K and K', and so

$$\rho(x,y) = G_K\left(\frac{U(x) - U(y)}{d_{CDF}(x,y)}\right) = G_{K'}\left(\frac{U(x) - U(y)}{d_{CDF}(x,y)}\right)$$

and since $\frac{U(x)-U(y)}{d_{CDF}(x,y)} = \frac{\alpha - \gamma}{\alpha + \gamma}$, for any $r \in [-1,1]$ we can choose $\alpha, \gamma \geq 0$ such that $\frac{U(x)-U(y)}{d_{CDF}(x,y)} = r$, we must have $G_K = G_{K'}$.

Finally, to show uniqueness, suppose, (G, β) and (G', u') both represent ρ. Define the stochastic preference relation $\succeq$ as before. Since G and G' are both increasing and symmetric around 0, $U(x) = \sum_s f_x(w)u(w)$ and $U'(x) = \sum_s f_x(w)u'(w)$ both represent $\succeq$, which satisfies the vNM axioms, we can invoke vNM to conclude that there exists $C > 0$, $b \in \mathbb{R}$ such that $u' = Cu + b$. This in turn implies that for all $x, y \in X$, we have

$$G\left(\frac{\sum_s (f_x(w)u(w) - f_y(w)u(w))}{\int_0^1 u(F_x^{-1}(q)) - u(F_y^{-1}(q))|\, dq}\right) = G'\left(\frac{\sum_s (f_x(w)u'(w) - f_y(w)u'(w))}{\int_0^1 u'(F_x^{-1}(q)) - u'(F_y^{-1}(q))|\, dq}\right)$$
$$= G'\left(\frac{\sum_s (f_x(w)u(w) - f_y(w)u(w))}{\int_0^1 u(F_x^{-1}(q)) - u(F_y^{-1}(q))|\, dq}\right)$$

Now consider $x, y \in X$ with

$$x = \begin{cases} w_b & \text{w.p. } 1 \end{cases} \qquad y = \begin{cases} w_c & \text{w.p. } \dfrac{\alpha/(u(w_b) - u(w_c))}{\alpha/(u(w_b) - u(w_c)) + \gamma/(u(w_a) - u(w_b))} \\ w_a & \text{w.p. } \dfrac{\gamma/(u(w_a) - u(w_b))}{\alpha/(u(w_b) - u(w_c)) + \gamma/(u(w_a) - u(w_b))} \end{cases}$$

since $\frac{U(x)-U(y)}{d_{CDF}(x,y)} = \frac{\alpha-\gamma}{\alpha+\gamma}$, for any $r \in [-1, 1]$ we can choose $\alpha, \gamma \geq 0$ such that $\frac{U(x)-U(y)}{d_{CDF}(x,y)} = r$, we must have $G' = G$.

□

Proof of Theorem 5.

For $x, y \in X$, $a, b \in \mathbb{R}$, define $ax + by \in X$ to be the payoff stream with the payoff function $am_x + bm_y$. Let $\phi^\tau \in X$ be the payoff stream that pays off 1 at time τ and 0 otherwise. We start by observing a Lemma.

Lemma 18. *Suppose $U : X \to \mathbb{R}$ is linear. Then there exists $d : [0, \infty) \to \mathbb{R}$ such that $U(x) = \sum_t d(t)m_x(t)$.*

Proof. Let $d : [0, \infty) \to \mathbb{R}$ satisfying $d(t) = U(\phi^t)$. Take any $x \in X$. Note that $x = \sum_{t \in T_x} m_x(t)\phi^t$, and so inductive application of linearity implies $U(x) = \sum_t d(t)m_x(t)$ as desired. □

Necessity of the axioms is immediate from the definitions; we now show sufficiency. Let $\succeq$ denote the complete binary relation on X induced by ρ. By Moderate Transitivity, $\succeq$ is transitive. Since ρ satisfies Continuity and Independence, by Theorem 8 in Herstein and Milnor (1953), $\succeq$ is represented by a linear $U : X \to \mathbb{R}$, and Lemma 18 in turn implies the existence of a $d : [0,\infty) \to \mathbb{R}$ such that $U(x) = \sum_t d(t)m_x(t)$. Dominance implies that $d(t)$ is positive and strictly decreasing. Extend d to $[0,\infty) \cup \{+\infty\}$ by taking $d(\infty) = 0$.

Fix any $t^a, t^b, t^c, t^d \in [0,\infty)$, $t^a < t^b < t^c < t^d$; we have $d(t^a) < d(t^b) < d(t^c) < d(t^d)$. Now consider any $x, y \in X$. Let $T = \{0, t^a, t^b, t^c, t^d\} \cup T_x \cup T_y$, and enumerate $T \cup \{\infty\}$ in increasing order by $\{t_1, t_2, ..., t_n, t_{n+1}\}$; we have $d(t_1) < d(t_2) < ... < d(t_{n+1})$. Let $X(T) = \{x \in X : T_x \subseteq T\}$ denote the set of payoff flows with support in T. Note that all $w \in X(T)$ corresponds to a unique $\tilde{w} \in \mathbb{R}^n$ satisfying $\tilde{w}_k = M_x(t_k)(d(t_k) - d(t_{k+1}))$. Denote by $\tilde{\rho}$ the induced preference on $\mathbb{R}^n$ satisfying $\tilde{\rho}(\tilde{x}, \tilde{y}) = \rho(x, y)$.

Claim 1. $\tilde{\rho}(\tilde{x}, \tilde{y}) \geq 1/2$ iff $\sum_k \tilde{x}_k \geq \sum_k \tilde{y}_k$. $\tilde{\rho}$ satisfies M1-M5.

Proof. Note that since $\sum_k \tilde{w}_k = \sum_t d(t)m_w(t)$ for all $w \in X(T)$, we have $\sum_k \tilde{x}_k \geq \sum_k \tilde{y}_k \iff \sum_t d(t)m_x(t) \geq \sum_t d(t)m_y(t) \iff \rho(x,y) \geq 1/2 \iff \tilde{\rho}(\tilde{x}, \tilde{y}) \geq 1/2$.

It is immediate that $\tilde{\rho}$ inherits Continuity, Linearity, and Moderate Stochastic Transitivity from ρ. Dominance follows from the fact that for all $x, y \in X(T)$, $M_x(t) \geq M_y(t)$ for all t if and only if $\tilde{x}_k \geq \tilde{y}_k$ for all k.

Finally, to see that $\tilde{\rho}$ satisfies Simplification, take any $\tilde{x}, \tilde{y} \in \mathbb{R}^n$ with $\tilde{\rho}(\tilde{x}, \tilde{y}) \geq 1/2$ and $i \neq j$, and consider $\tilde{x}'$ satisfying $\tilde{x}'_i = \tilde{y}_i$, $\tilde{x}'_k = \tilde{x}_k$ for $k \neq i, j$, and with $\tilde{\rho}(\tilde{x}', \tilde{x}) = 1/2$. By construction, we have $\rho(x, y) \geq 1/2$, $\rho(x', x) \geq 1/2$. Since $m_x(t), m_y(t) \neq 0$ for finitely many t, there exists $\eta \in \mathbb{R}$ such that $m_x(t) + \eta \neq 0$ and $m_y(t) + \eta \neq 0$ for all t. Let $z \in X(T)$ denote the payoff flow with $m_z(t) = \eta$ for all $t \in T$, and $m_z(t) = 0$ otherwise. Define $\hat{x}, \hat{x}', \hat{y} \in X$ by $\hat{x} = x + z$, $\hat{x}' = x' + z$, $\hat{y} = y + z$. By Linearity of ρ, we have $\rho(\hat{x}, \hat{y}) \geq 1/2$, $\rho(\hat{x}', \hat{x}) \geq 1/2$. Note that by construction, $T_{\hat{x}} = T_{\hat{x}'} = T_{\hat{y}} = \{t_1, ..., t_n\}$, and so the support of $\hat{x}'$ is contained in $T_{\hat{x}} \cup T_{\hat{y}}$. Furthermore, $\tilde{x}'_i = \tilde{y}_i$ implies $M_{\hat{x}'}(t_i) = M_{\hat{y}}(t_i)$, and $\tilde{x}'_k = \tilde{y}_k$ for all $k \neq i, j$ implies $M_{\hat{x}'}(t) = M_{\hat{x}}(t)$ for all $t \in T_{\hat{x}} \cup T_{\hat{y}} / \{t_i, t_j\}$, and so since ρ satisfies Simplification, we have $\rho(\hat{x}', \hat{y}) \geq \rho(\hat{x}, \hat{y})$. Linearity of ρ then implies that $\rho(x', y) \geq \rho(x, y)$, and so by definition of $\tilde{\rho}$ we have $\tilde{\rho}(\tilde{x}', \tilde{y}) \geq \tilde{\rho}(\tilde{x}, \tilde{y})$ as desired. □

Using Claim 1, Theorem 1 then implies that there exists a continuous, strictly increasing

$G : [-1,1] \to [0,1]$, symmetric around 0, such that for all $x, y \in X(T)$ $\tilde{x}, \tilde{y} \in \mathbb{R}^n$, we have

$$\begin{aligned}\rho(x,y) &= \tilde{\rho}(\tilde{x},\tilde{y}) \\ &= G\left(\frac{\sum_k (\tilde{x}_k - \tilde{y}_k)}{\sum_k |\tilde{x}_k - \tilde{y}_k|}\right) \\ &= G\left(\frac{U(x)-U(y)}{d_{CPF}(x,y)}\right)\end{aligned}$$

Let $\mathcal{T} = \{T \subseteq [0,\infty) : |T| < \infty, \{0, t^a, t^b, t^c, t^d\} \subseteq T\}$. The above implies that for all $T \in \mathcal{T}$, there exists a continuous, strictly increasing $G_T : [-1,1] \to [0,1]$, symmetric around 0 such that for any $x, y \in X(T)$,

$$\rho(x,y) = G_T\left(\frac{U(x)-U(y)}{d_{CPF}(x,y)}\right)$$

Since for any $x, y \in X$, there exists some $T \in \mathcal{T}$ such that $x, y \in X(T)$, all that remains to show that All that remains is to show that $G_T = G_{T'}$ for any $T, T' \in \mathcal{T}$. To see this, fix any $T, T' \in \mathcal{T}$, and consider $x, y \in X$ with

$$m_x(t) = \begin{cases} \alpha/(d(t_a)-d(t_b)) & t = t_a \\ \gamma/(d(t_b)-d(t_c)) & t = t_c \\ 0 & \text{otherwise} \end{cases} \qquad m_y(t) = \begin{cases} \alpha/(d(t_a)-d(t_b)) + \gamma/(d(t_b)-d(t_c)) & t = t_b \\ 0 & \text{otherwise} \end{cases}$$

for some $\alpha \geq 0$, $\gamma \geq 0$. Note that x, y belong to both T and T', and so we have

$$\rho(x,y) = G_T\left(\frac{U(x)-U(y)}{d_{CPF}(x,y)}\right) = G_{T'}\left(\frac{U(x)-U(y)}{d_{CPF}(x,y)}\right)$$

and since $\frac{U(x)-U(y)}{d_{CPF}(x,y)} = \frac{\alpha-\gamma}{\alpha+\gamma}$, for any $r \in [-1,1]$ we can choose $\alpha, \gamma \geq 0$ such that $\frac{U(x)-U(y)}{d_{CPF}(x,y)} = r$, we must have $G_T = G_{T'}$.

Finally, to show uniqueness, suppose (G, d) and (G', d') both represent ρ. Define the stochastic preference relation $\succeq$ as before. Since G, G' are both strictly increasing, symmetric around 0, both $U(x) = \sum_t d(t) m_x(t)$ and $U'(x) = \sum_t d'(t) m_x(t)$ both represent $\succeq$. Since $d \geq 0$ and and d, d' are both strictly decreasing, we have $d(0), d'(0) > 0$. Fix any $t \in (0,\infty)$, and let $\lambda_t = d(t)/d(0)$. By construction, $U(\phi^t) = U(\lambda_t \phi^0)$, and so $\phi^t \sim \lambda_t \phi^0$. Since U' also represents $\succeq$, we have $U'(\phi^t) = U'(\lambda_t \phi^0) \implies d'(t) = \lambda_t d'(0)$, and so $d'(t) = C d(t)$

for all $t \in [0, \infty)$, where $C = d'(0)/d(0) > 0$. This in turn implies that for all $x, y \in X$, $\{t_0, t_1, ..., t_n\}$ containing $\{0, \infty\} \cup T_x \cup T_y$,

$$\begin{aligned} G\left(\frac{U(x)-U(y)}{d_{CPF}(x,y)}\right) &= G'\left(\frac{\sum_k (d'(t_k)m_x(t_k) - d'(t_k)m_y(t_k))}{\sum_k |M_x(t_k) - M_y(t_k)|(d'(t_k) - d'(t_{k+1}))}\right) \\ &= G'\left(\frac{\sum_k (d(t_k)m_x(t_k) - d(t_k)m_y(t_k))}{\sum_k |M_x(t_k) - M_y(t_k)|(d(t_k) - d(t_{k+1}))}\right) \\ &= G'\left(\frac{U(x)-U(y)}{d_{CPF}(x,y)}\right) \end{aligned}$$

Consider $x, y \in X$ with

$$m_x(t) = \begin{cases} \alpha/(d(t_a) - d(t_b)) & t = t_a \\ \gamma/(d(t_b) - d(t_c)) & t = t_c \\ 0 & \text{otherwise} \end{cases} \qquad m_y(t) = \begin{cases} \alpha/(d(t_a) - d(t_b)) + \gamma/(d(t_b) - d(t_c)) & t = t_b \\ 0 & \text{otherwise} \end{cases}$$

for some $\alpha \geq 0$, $\gamma \geq 0$. Since $\frac{U(x)-U(y)}{d_{CPF}(x,y)} = \frac{\alpha-\gamma}{\alpha+\gamma}$, for any $r \in [-1, 1]$ we can choose $\alpha, \gamma \geq 0$ such that $\frac{U(x)-U(y)}{d_{CPF}(x,y)} = r$, we must have $G' = G$.

$\square$

F.2 Other Results

Proof of Proposition 6

Suppose $(G, P, (u_E)_{E \in P})$ and $(\tilde{G}, \tilde{P}, (\tilde{u}_E)_{E \in \tilde{P}})$ represent ρ. We will first show that $P = \tilde{P}$. Fix $E \in P$. It suffices to show that $E \in \tilde{P}$.

First, show that there must exist $\tilde{E} \in \tilde{P}$ such that $E \subseteq \tilde{E}$. Toward a contradiction, suppose not: there then exists indices $i, j \in E$ such that $i \in \tilde{E}$ and $j \in \tilde{E}'$ for $\tilde{E}, \tilde{E}' \in \tilde{P}$, $\tilde{E} \neq \tilde{E}'$.

Since $\tilde{u}_{\tilde{E}}$ is non-trivial, there exists $w_{-i} \in X_{\tilde{E}\setminus\{i\}}$ such that for some $x_i, x_i' \in X_i$, $\tilde{u}_{\tilde{E}}(x_i, w_{-i}) \neq \tilde{u}_{\tilde{E}}(x_i', w_{-i})$. Since $\tilde{u}_{\tilde{E}}$ is continuous, the mapping $x_i \mapsto \tilde{u}(x_i, w_{-i})$, which we denote by v_i, is continuous, and since X_i is connected and separable, the codomain of this mapping is a non-trivial interval. By a similar argument, there exists $z_{-j} \in X_{\tilde{E}'\setminus\{j\}}$ such that the codomain of the mapping $x_j \mapsto \tilde{u}_{\tilde{E}'}(x_j, w_{-j})$, which we denote by v_j, is a non-trivial interval.

The above implies that there exists $a, b, c \in X_i$, $\alpha, \beta \in X_j$, such that $v_i(a) > v_i(b) > v_i(c)$,

$v_j(\alpha) > v_j(\beta)$, and $v_i(a) - v_i(b) > v_j(\alpha) - v_j(\beta)$. Fixing any $h \in X_{(\tilde{E}\cup\tilde{E}')^c}$, define $x, y \in X$ by

$$x_k = \begin{cases} (w_{-i})_k & k \in \tilde{E} \setminus \{i\} \\ (z_{-j})_k & k \in \tilde{E}' \setminus \{j\} \\ b & k = i \\ \beta & k = j \\ h_k & \text{otherwise} \end{cases} \qquad x'_k = \begin{cases} (w_{-i})_k & k \in \tilde{E} \setminus \{i\} \\ (z_{-j})_k & k \in \tilde{E}' \setminus \{j\} \\ a & k = i \\ \beta & k = j \\ h_k & \text{otherwise} \end{cases} \qquad y_k = \begin{cases} (w_{-i})_k & k \in \tilde{E} \setminus \{i\} \\ (z_{-j})_k & k \in \tilde{E}' \setminus \{j\} \\ c & k = i \\ \alpha & k = j \\ h_k & \text{otherwise} \end{cases}$$

By construction, we have $x'_k = x_k = y_k$ for all $k \neq i, j$, and $\tilde{u}_{\tilde{E}}(x'_{\tilde{E}}) > \tilde{u}_{\tilde{E}}(x_{\tilde{E}}) > \tilde{u}_{\tilde{E}}(y_{\tilde{E}})$ and $\tilde{u}_{\tilde{E}'}(x'_{\tilde{E}'}) = \tilde{u}_{\tilde{E}'}(x_{\tilde{E}'}) < \tilde{u}_{\tilde{E}'}(y_{\tilde{E}'})$, where $\tilde{u}_{\tilde{E}'}(y_{\tilde{E}'}) - \tilde{u}_{\tilde{E}'}(x_{\tilde{E}'}) < \tilde{u}_{\tilde{E}}(x_{\tilde{E}}) - \tilde{u}_{\tilde{E}}(y_{\tilde{E}})$.

Since $(\tilde{G}, \tilde{P}, (\tilde{u}_E)_{E\in\tilde{P}})$ represents ρ, we have

$$\rho(x, y) = G'\left(\frac{\tilde{u}_{\tilde{E}}(x_{\tilde{E}}) - \tilde{u}_{\tilde{E}}(y_{\tilde{E}}) + \tilde{u}_{\tilde{E}'}(y_{\tilde{E}'}) - \tilde{u}_{\tilde{E}'}(x_{\tilde{E}'})}{|\tilde{u}_{\tilde{E}}(x_{\tilde{E}}) - \tilde{u}_{\tilde{E}}(y_{\tilde{E}})| + |\tilde{u}_{\tilde{E}'}(y_{\tilde{E}'}) - \tilde{u}_{\tilde{E}'}(x_{\tilde{E}'})|}\right) > 1/2$$

and we also have $\rho(x', y) > \rho(x, y)$. Since $(G, P, (u_E)_{E\in P})$ also represents ρ, we have

$$\rho(x, y) = G\left(\frac{u_E(x_E) - u_E(y_E)}{|u_E(x_E) - u_E(y_E)|}\right)$$

and since $\rho(x, y) > 1/2$, it must be the case that $u_E(x_E) - u_E(y_E) > 0$ and so $\rho(x, y) = G(1)$. But this contradicts the fact that $\rho(x', y) > \rho(x, y)$, since by definition G attains its maximal value at 1.

Now, we show that there cannot exist $\tilde{E} \in \tilde{P}$ such that E is a strict subset of $\tilde{E}$. To see this, suppose that E is a strict subset of $\tilde{E}$: there then exists $i, j \in \tilde{E}$ such that $i \in E$ and $j \in E' \in P$, for $E \neq E'$. But by an argument analogous to the one above, this cannot be the case, and so we have a contradiction. We have therefore shown that for all $E \in P$, there exists $\tilde{E} \in \tilde{P}$ such that $E = \tilde{E}$, and so $P = \tilde{P}$.

By relabeling each $E \in P$ as an attribute, the above implies that ρ has additively separable L_1 complexity representations $(G, (u_E)_{E\in P})$ and $(\tilde{G}, (\tilde{u_E})_{E\in P})$. By Theorem 3, $G = G'$, and there exists $C > 0$, $b_E \in \mathbb{R}$ such that for each $E \in P$, $\tilde{u}_E = Cu_E + b_E$.

□

Proof of Proposition 7

Note that since H is strictly increasing,

$$\max_{g\in\Gamma(x,y)} \tau_{xy}^{L1}(g) = \max_{g\in\Gamma(x,y)} H\left(\frac{|EU(x)-EU(y)|}{\sum_{w_x,w_y}|g(w_x,w_y)(u(w_x)-u(w_y))|}\right)$$
$$= H\left(\frac{|EU(x)-EU(y)|}{\min\limits_{g\in\Gamma(x,y)}\sum_{w_x,w_y}|g(w_x,w_y)(u(w_x)-u(w_y))|}\right)$$

Let $\tilde{x}$ and $\tilde{y}$ denote the utility-valued lotteries induced by x and y, defined by the quantile functions $F_{\tilde{x}}^{-1}(q) = u(F_x^{-1}(q))$ and $F_{\tilde{y}}^{-1}(q) = u(F_y^{-1}(q))$ for all $q \in [0,1]$. Note that

$$\begin{aligned}\min_{g\in\Gamma(x,y)}\sum_{w_x,w_y}|g(w_x,w_y)(u(w_x)-u(w_y))| &= \min_{g\in\Gamma(\tilde{x},\tilde{y})}\sum_{w_x,w_y} g(w_x,w_y)|(w_x-w_y)| \\ &= \int_{-\infty}^{\infty}|F_{\tilde{x}}(w)-F_{\tilde{y}}(w)|\,dw \\ &= \int_0^1 |F_{\tilde{x}}^{-1}(q)-F_{\tilde{y}}^{-1}(q)|\,dq \\ &= d_{CDF}(x,y)\end{aligned}$$

Where the second equality follows from Vallender (1974), since $\min\limits_{g\in\Gamma(\tilde{x},\tilde{y})}\sum_{w_x,w_y}|g(w_x,w_y)(w_x-w_y)|$ is the 1-Wassertein metric between the distributions $F_{\tilde{x}}$ and $F_{\tilde{y}}$, the third equality follows from a change of variables, and the final equality follows from the definition of $\tilde{x}$, $\tilde{y}$.

□

Proof of Proposition 8

Note that since H is strictly increasing,

$$\max_{b\in B(x,y)} \tau_{xy}^{L1}(b) = \max_{b\in B(x,y)} H\left(\frac{|DU(x)-DU(y)|}{\sum_{t_x,t_y}|b(t_x,t_y)(d(t_x)-d(t_y))|}\right)$$
$$= H\left(\frac{|DU(x)-DU(y)|}{\min\limits_{b\in B(x,y)}\sum_{t_x,t_y}|b(t_x,t_y)(d(t_x)-d(t_y))|}\right)$$

All that remains is to show that for $d^b_{L1}(x,y) \equiv \sum_{t_x,t_y} |b(t_x,t_y)d(t_x) - d(t_y)|$, we have $\min_{b \in B(x,y)} d^b_{L1}(x,y) = d_{CPF}(x,y)$.

Without loss, normalize $d(0) = 1$, and fix any x, y. Let $\overline{w} = \sum_t m_x(t) + \sum_t m_y(t)$ denote the total payoff delivered by both x and y. Let $\overline{B}(x,y)$ contain all $b \in B(x,y)$ satisfying $b(t_x,t_y) > 0$ for all t_x, t_y. Note that this implies that for all $b \in \overline{B}(x,y)$, we have $\sum_{t_x,t_y} b(t_x,t_y) \leq \overline{w}$. Since x and y have positive payouts, we have

$$\max_{b \in B_{x,y}} d^b_{L1}(x,y) = \max_{b \in \overline{B}_{x,y}} d^b_{L1}(x,y)$$

We will now show that $\max_{b \in \overline{B}_{x,y}} d^b_{L1}(X,Y) = d_{CPP}(x,y)$. For all $b \in \overline{B}(X,Y)$, consider a joint density $\tilde{b}$ over $[0,1]^2$ with mass function satisfying

$$\tilde{b}(w_x,w_y) = \begin{cases} b(d^{-1}(w_x), d^{-1}(w_y)/\overline{w} & w_x \neq 0 \text{ or } w_y \neq 0 \\ 1 - \sum_{\{(t_x,t_y):\neg(t_x=\infty,t_y=\infty)\}} b(t_x,t_y)/\overline{w} & w_x = w_y = 0 \end{cases}$$

Note that $\tilde{b}$ is well-defined since $b(t_x,t_y) > 0$ for all t_x, t_y and $\sum_{t_x,t_y} b(t_x,t_y)/\overline{w} \leq 1$ by construction.

Let $\tilde{b}_x$ and $\tilde{b}_y$ denote the marginal distributions of $\tilde{b}$. Note that for all $t \in [0,\infty)$, we have

$$\begin{aligned} \tilde{b}_x(d(t)) &= \sum_{w_y} \tilde{b}(d(t), w_y) \\ &= \sum_{t_y} \tilde{b}(d(t), d(t_y))/\overline{w} \\ &= \sum_{t_y} b(t, t_y)/\overline{w} \\ &= m_x(t)/\overline{w} \end{aligned}$$

where the third equality follows from the fact that $\sum_{t_y} b(t,t_y) = m_x(t)$ for all $t \in [0,\infty)$, and so

$$\tilde{b}_x(w) = h_x(w) \equiv \begin{cases} m_x(d^{-1}(w))/\overline{w} & w \in (0,1] \\ 1 - \sum_t m_x(t)/\overline{w} & w \in 0 \end{cases}$$

A similar argument implies that

$$\tilde{b}_y(w) = h_y(w) \equiv \begin{cases} m_y(d^{-1}(w))/\overline{w} & w \in (0,1] \\ 1 - \sum_t m_y(t)/\overline{w} & w \in 0 \end{cases}$$

Let $\tilde{B}(x,y)$ denote the set of joint densities $g(w_x, w_y)$ over $[0,1]^2$ with marginals given by $g_x = h_x$ and $g_y = h_y$. The above implies that for all $b \in \overline{B}(x,y)$, $\tilde{b} \in \tilde{B}(x,y)$. We will now show that for all $g \in \tilde{B}(x,y)$, there exists $b \in \overline{B}(x,y)$ such that $\tilde{b} = g$.

Fix any $g \in \tilde{B}(x,y)$, and define $b : \mathbb{R}_+ \cup \{+\infty\} \times \mathbb{R}_+ \cup \{+\infty\} \to \mathbb{R}$ by

$$b(d^{-1}(w_x), d^{-1}(w_y)) = \begin{cases} g(w_x, w_y) \cdot \overline{w} & w_x \neq 0 \text{ or } w_y \neq 0 \\ 0 & w_x = w_y = 0 \end{cases}$$

for all $w_x, w_y \in [0,1]^2$. By construction, $\sum_{t_x, t_y} b(t_x, t_y) \leq \overline{w}$ and $b(t_x, t_y) > 0$. Furthermore, for all $t \in [0, \infty)$ we have

$$\begin{aligned} \sum_{t_y} b(t, t_y) &= \sum_{w_y} b(t, d^{-1}(w_y)) \\ &= \sum_{w_y} g(d(t), w_y) \cdot \overline{w} \\ &= h_x(d(t)) \cdot \overline{w} \\ &= m_x(t) \end{aligned}$$

where the third equality follows from the fact that $g_x = h_x$ and the last equality follows from the definition of h_x. We similarly have $\sum_{t_x} b(t_x, t) = my(t)$ for all $t \in [0, \infty)$, and so $b \in \overline{B}(x,y)$. Note that by construction, $\tilde{b} = g$ as desired. Now since

$$\begin{aligned} d_{L1}^b(x,y) &= \sum_{t_x, t_y} b(t_x, t_y) |d(t_x) - d(t_y)| \\ &= \overline{w} \sum_{w_x, w_y} \tilde{b}(w_x, w_y) |w_x - w_y| \end{aligned}$$

the fact that for any $b \in \overline{B}(x,y)$, $\tilde{b} \in \tilde{B}(x,y)$ and that for any $g \in \tilde{B}(x,y)$, there exists

$b \in \overline{B}(x,y)$ s.t. $\tilde{b} = g$ implies that

$$\begin{aligned}
\min_{b \in \overline{B}(x,y)} d^b_{L1}(x,y) &= \min_{g \in \tilde{B}(x,y)} \overline{w} \sum_{w_x, w_y} g(w_x, w_y)|w_x - w_y| \\
&= \overline{w} \int_0^1 |H_x(w) - H_y(w)| \, dw
\end{aligned}$$

where the second line follows from Vallender (1974), for H_x and H_y the CDFs of h_x, h_y. Enumerate the elements of T_{xy} by $0 = t_0, t_1, ..., t_n = \infty$ and let $w_k = d(t_k)$ for all $k = 0, 1, ..., n$. Note that for all $k = 1, ..., n$,

$$\begin{aligned}
H_x(w_k) &= \sum_{j=k}^{n-1} m_x(d^{-1}(w_j))/\overline{w} + 1 - \sum_{j=1}^{n-1} m_x(t_j)/\overline{w} \\
&= 1 - \sum_{j=1}^{k-1} m_x(t_j)/\overline{w} \\
&= 1 - M_x(t_{k-1})/\overline{w}
\end{aligned}$$

By a similar argument for H_y, we have

$$H_x(w_k) = \begin{cases} 1 - M_x(t_{k-1})/\overline{w} & k \geq 1 \\ 1 & k = 0 \end{cases} \quad H_y(w_k) = \begin{cases} 1 - M_y(t_{k-1})/\overline{w} & k \geq 1 \\ 1 & k = 0 \end{cases}$$

We therefore have

$$\begin{aligned}
\min_{b \in \overline{B}(x,y)} d^b_{L1}(x,y) &= \overline{w} \sum_{k=1}^{n} |H_x(w_k) - H_y(w_k)|(w_{k-1} - w_k) \\
&= \sum_{k=1}^{n} |M_x(t_{k-1}) - M_y(t_{k-1})|(d(t_{k-1}) - d(t_k)) \\
&= d_{CPF}(x,y)
\end{aligned}$$

as desired.

$\square$

Proof of Proposition 9

Suppose a multinomial choice rule ρ is represented by (Q, v, τ) and (Q', v', τ'). With some abuse of notation, let ρ also denote the binary choice rule induced by the restriction of ρ to binary menus.

Let $\succeq$ denote the stochastic order induced by ρ. Since ρ is represented by (Q, v, τ), we have $\rho(x, y) = \Phi(\text{sgn}(v(x) - v(y))\tau(x, y))$, and so $x \succeq y$ iff $v(x) \geq v(y)$. Similarly, since ρ is represented by (Q', v', τ'), $x \succeq y$ iff $v'(x) \geq v'(y)$. This implies that for any x, y, we have $v(x) = v(y) \iff x \sim y \iff v'(x) = v'(y)$, and so the transformation $\phi : v(X) \to \mathbb{R}$ satisfying $\phi(v(x)) = v'(x)$ for all $x \in X$ is well defined. To see that ϕ is strictly increasing, suppose not; there exists $x, y \in X$ such that $v(x) > v(y)$ but $\phi(v(x)) \leq \phi(v(y))$; the former implies that $x \succ y$ but the latter implies that $y \succeq x$, a contradiction.

To see that $\tau = \tau'$, fix any $(x, y) \in \mathcal{D}$. First consider the case where $v(x) = v(y)$; by definition of τ, $\tau(x, y) = 0$. But since $v(x) = v(y) \implies v'(x) = v'(y)$, we also have $\tau'(x, y) = 0$. Now consider the case where $v(x) \neq v(y)$; without loss, assume $v(x) > v(y)$. By the above result, we have $\text{sgn}(v(x) - v(y)) = \text{sgn}(v'(x) - v'(y)) = 1$, which in turn implies that $\rho(x, y) = \Phi(\tau(x, y)) = \Phi(\tau'(x, y))$. Since Φ is strictly increasing, we have $\tau(x, y) = \tau'(x, y)$, and so $\tau = \tau'$ as desired.

□

Proof of Proposition 10

Consider the extension of a binary choice rule ρ to a multinomial choice rule $\rho(x,A)$ satisfying $\sum_{x \in A} \rho(x,A) = 1$ in every finite menu A. Note that ρ describes a mapping from finite subsets of $\mathbb{R}^n$ to a probability measure on the Borel sigma-algebra in $\mathbb{R}^n$. Endow the set of finite subsets of $\mathbb{R}^n$ with the topology induced by the Hausdorff metric, and endow the set of probability measures on the Borel sigma-algebra with the topology of weak convergence. We now state the Gul and Pesendorfer (2006) postulates (henceforth, GP postulates). Say that ρ is *continuous* when this mapping is continuous. Say that ρ is *monotone* if $\rho(x,A) \geq \rho(x,B)$ whenever $A \subseteq B$, and that ρ is *linear* if for all $x \in$, $y \in \mathbb{R}^n$, $\alpha \in (0, 1)$, $\rho(x,A) = \rho(\alpha x + (1 - \alpha y), \{\alpha z + (1 - \alpha)y : z \in A\})$. Say that ρ is *extreme* if $\rho(x,A) > 0$ implies that x is an extreme point of A.

Now consider a binary choice rule ρ with an L_1-complexity representation (β, G), where $G(1) = G(-1) = 1$. Note that on its domain, ρ satisfies the GP postulates of continuity, linearity, and extremeness, as in any binary menu both options are extreme points. Since the GP postulate of monotonicity makes no restrictions in binary choice behavior, ρ also satisfies this property on its domain. Therefore, there exists an extension of ρ to a multinomial choice rule that is conitnuous, monotone, linear, and extreme. By Theorem 3 of Gul and Pesendorfer (2006), there exists a random vector $\tilde{\beta}$ such that

$$\rho(x,A) = \mathbb{P}\left\{\sum_k \tilde{\beta}_k x_k \geq \sum_k \tilde{\beta}_k y_k \, \forall \, y \in A\right\}$$

which in turn implies that for all $(x, y) \in \mathcal{D}$,

$$\rho(x,y) = \mathbb{P}\left\{\sum_k \tilde{\beta}_k x_k \geq \sum_k \tilde{\beta}_k y_k\right\}$$

To see that $\mathbb{P}\left\{\text{sgn}(\tilde{\beta}_k) = \text{sgn}(\beta_k)\right\} = 1$ for all k, suppose not; let $y = \vec{0}$ and $x \in \mathbb{R}^n$ such that $x_k = \text{sgn}(\beta_k)$ and $x_j = 0$ for all $j \neq k$. Since $G(1) = 1$ and $\beta_k x_k \geq \beta_k y_k$ for all k, we have $\rho(x, y) = 1$. However, $\mathbb{P}\left\{\text{sgn}(\tilde{\beta}_k) = \text{sgn}(\beta_k)\right\} \neq 1$ implies that $\rho(x, y) < 1$, a contradiction. $\square$

Proof of Proposition 11

Suppose that ρ has an linear differentiation representation (β, Σ, G) and that at least 3 attributes are non-null; without loss, we assume that attributes $k = 1, 2, 3$ are non-null.

Let $\tilde{\rho}$ denote the binary choice rule on $\mathbb{R}^2$ defined by the restriction of ρ to the first two dimensions, i.e. $\tilde{\rho}(\tilde{x}, \tilde{y}) = \rho((\tilde{x}, 0, ..., 0), (\tilde{y}, 0, ..., 0))$ for all $\tilde{x}, \tilde{y} \in \mathbb{R}^2$, $\tilde{x} \neq \tilde{y}$; it is immediate from the definition that $\tilde{\rho}$ has an linear differentiation representation with parameters $(\tilde{\beta}, \tilde{\Sigma}, G)$, where $\tilde{\beta} = (\beta_1, \beta_2)$ and $\tilde{\Sigma}$ is the submatrix formed from the first 2 rows and columns of Σ. Furthermore, since attributes 1 and 2 are non-null, $\tilde{\beta}_1, \tilde{\beta}_2 \neq 0$.

Fix any $\tilde{y} \in \mathbb{R}^n$, and define $B = \{\tilde{x} \in \mathbb{R}^2 : \tilde{\beta}'(\tilde{x} - \tilde{y}) = 1\}$. Note that $\arg\max_{\tilde{x}' \in B} \tilde{\rho}(\tilde{x}', \tilde{y})$ has a unique maximizer, which we denote by $\tilde{x}$: to see this, note that Proposition 1 of He and Natenzon (2023b) implies that if $\tilde{x} \in \arg\max_{\tilde{x}' \in B} \tilde{\rho}(\tilde{x}', \tilde{y})$, then $\tilde{x} - \tilde{y} = \alpha \tilde{\Sigma}^{-1} \tilde{\beta}$ for some $\alpha \neq 0$; since $\tilde{\beta}'(\tilde{x} - \tilde{y}) = 1$, it must be the case that $\alpha = 1/\tilde{\beta}' \tilde{\Sigma}' \tilde{\beta}$ and so $\tilde{x} = \tilde{y} + \frac{1}{\tilde{\beta}' \tilde{\Sigma}' \tilde{\beta}} \tilde{\Sigma}^{-1} \tilde{\beta}$.

Take any $\tilde{w} \neq \tilde{x}$ such that $\tilde{w} \in B$ and $\text{sgn}(\tilde{w}_k) = \text{sgn}(\tilde{\beta}_k)$ for $k = 1, 2$, and define $x, w, y \in \mathbb{R}^n$ where $x = (\tilde{x}, 0, ..., 0)$, $w = (\tilde{w}, 0, ..., 0)$, $y = (\tilde{y}, 0, ..., 0)$. Since $\tilde{x}$ is the unique maximizer of $\arg\max_{\tilde{x}' \in B} \tilde{\rho}(\tilde{x}', \tilde{y})$, we have $\tilde{\rho}(\tilde{w}, \tilde{y}) < \tilde{\rho}(\tilde{x}, \tilde{y})$, which in turn implies $\rho(w, y) < \rho(x, y)$. Furthermore, since by construction we have $\text{sgn}(x_k) = \text{sgn}(\beta_k)$ for $k = 1, 2$ and $x_k = 0$ for all $k > 2$, and since $\beta_1, \beta_2 \neq 0$, we have $w >_D y$.

Note that if $x \not>_D y$, we are done. If instead $x >_D y$, define $x'(\epsilon) \in \mathbb{R}^n$ by $x'(\epsilon) = (\tilde{x}_1, \tilde{x}_2, -\text{sgn}(\beta_3)\epsilon, 0, ..., 0)$; by continuity of ρ, there exists some $\epsilon > 0$ such that $\rho(w, y) <$

$\rho(x'(\epsilon), y)$. Furthermore, since $\beta_3 \neq 0$ as the third attribute is non-null, by construction we have $x'(\epsilon) \not>_D y$ and so we are done.

$\square$

Proof of Proposition 12

Suppose ρ has an L_1-complexity representation. Theorem 2 implies that ρ satisfies moderate transitivity and dominance with respect to $>_D$, and so Lemma 1 implies that ρ satisfies monotonicity with respect to $>_D$, which in turn implies weak monotonicity.

Now suppose that ρ has a linear differentiation representation (β, Σ, G) and suppose that at least two attributes are non-null; without loss, we take these attributes to be $k = 1, 2$.

Let $\tilde{\rho}$ denote the binary choice rule on $\mathbb{R}^2$ defined by the restriction of ρ to the first two dimensions, i.e. $\tilde{\rho}(\tilde{x}, \tilde{y}) = \rho((\tilde{x}, 0, \dots, 0), (\tilde{y}, 0, \dots, 0))$ for all $\tilde{x}, \tilde{y} \in \mathbb{R}^2$, $\tilde{x} \neq \tilde{y}$; it is immediate from the definition that $\tilde{\rho}$ has an linear differentiation representation with parameters $(\tilde{\beta}, \tilde{\Sigma}, G)$, where $\tilde{\beta} = (\beta_1, \beta_2)$ and $\tilde{\Sigma}$ is the submatrix formed from the first 2 rows and columns of Σ. Furthermore, since attributes 1 and 2 are non-null, $\tilde{\beta_1}, \tilde{\beta_2} \neq 0$.

Fix any $\tilde{y} \in \mathbb{R}^2$. Proposition 1 of He and Natenzon (2023b) implies that any $\tilde{x} \in \arg\max_{\tilde{x}'} \tilde{\rho}(\tilde{x}', \tilde{y})$ satisfies $\tilde{x} - \tilde{y} = \alpha \tilde{\Sigma}^{-1} \tilde{\beta}$ for some $\alpha \neq 0$; fix such a $\tilde{x}$. Since $\left\{\alpha \tilde{\Sigma}^{-1} \tilde{\beta}\right\}_{\alpha \in \mathbb{R}}$ traces a unique direction in $\mathbb{R}^2$, there exists $b_1, b_2 > 0$ such that for $b \equiv (\mathrm{sgn}(\beta_1) \cdot b_1, \mathrm{sgn}(\beta_2) \cdot b_2)$, we have $b \neq \alpha \tilde{\Sigma}^{-1} \tilde{\beta}$ for any $\alpha \neq 0$, which in turn implies that for $\tilde{x}' \equiv \tilde{x} + b$, $\tilde{x}' - \tilde{y} \neq \alpha \tilde{\Sigma}^{-1} \tilde{\beta}$ for any $\alpha \neq 0$, and so $\tilde{\rho}(\tilde{x}', \tilde{y}) < \tilde{\rho}(\tilde{x}, \tilde{y})$.

Now define $x, x', y \in \mathbb{R}^n$ by $x = (\tilde{x}, 0, \dots, 0)), x' = (\tilde{x}', 0, \dots, 0)), y = (y, 0, \dots, 0))$; by construction and the above, we have $\rho(x', y) < \rho(x, y)$. Also by construction, we have $x' >_D x$, and so ρ violates weak monotonicity.

G Performance Benchmarks

Following the procedure proposed in Fudenberg et al. (2022), we establish "completeness benchmarks" in each of our three domains by training flexible, non-parametric models to predict our outcomes of interest (i.e. choice rates). Then, we assess the completeness of our similarity-based complexity model as well as other canonical choice models. In this section, we describe the procedure used to train non-parametric benchmark models in each domain.

Overview. We use neural networks in each domain to fit highly flexible models of choice on binary choice data. The inputs to the network are problem fundamentals (attribute values in multi-attribute choice; payoffs and delays in intertemporal choice; payoffs and probabilities in lotteries) and a set of hand-coded transformations of problem fundamentals.

To get our final "best performing" model, we use linear regression to ensemble the neural network prediction with the predictions of several alternative models in a validation set. Finally, we use this ensembled predictor as our benchmark in a left-out test set. Because we want to use our full sample of problems in final analysis, we split each dataset into 10 equal-sized folds and train 10 separate fully out-of-sample predictors, one for each fold.

Neural Network Training. In each domain and for each training fold, we tune a neural network with 1 to 3 layers of hidden nodes. The set of hyperparameters we tune over are displayed in Table 1. We use a learning rate of 0.001 and an Adam optimizer. We initially experimented with alternative learning rates (including learning rate schedulers) and optimizers, but found that these options performed at least as well as alternatives on our data.

Hyperparameter	**Values**	**Meaning**
Number of layers	1, 2, 3	Number of linear layers included in the network
Nodes	8, 16, 32	Number of hidden nodes in each network layer
Random dropout	0.0, 0.2, 0.5	Fraction of nodes to randomly zero out
Batch size	8, 16, 32, 64	Number of observations network should handle at once
Number of epochs	100, 500, 1000	Number of training epochs

Table 1: Hyperparameter tuning grid for neural networks.

Given a training set (8 folds), a validation set (1 fold), and a test set (1 fold), we proceed by training a model with every possible combination of hyperparameters on the training set. [1] We then selected the "best" set of hyperparameters by evaluating the models' performance on the validation set. In particular, we select the model which minimizes negative log likelihood. Finally, we get test set predictions from the network trained with this "best" set of hyperparameters. We repeat this procedure 10 times to get a fully out-of-sample predictions for each choice in our data. For intertemporal choice and multi-attribute choice, we separately select hyperparameters on each training fold. In lottery choice, we expedite the training process by selecting hyperparameters only once (while getting out-of-sample predictions for the first fold) and then use this same set of hyperparameters to train the other 9 networks. We made this choice due to computational limitations, as there are over 10,000 lottery choice problems (10-20 times as many observations as our other domains) which

[1] In some domains, our grid search is based on a subset of the hyperparameter values from Table 1. We first did experimentation to cull clearly underperfoming hyperparameters from the search space.

makes training over a large hyperparameter grid 10 times costly.

Ensemble Procedure. After getting neural network predictions, we perform an ensembling step to produce our "completeness benchmark" predictive model. We always perform the ensembling step in the validation set so that the final test-set predictions are entirely out of sample. This means that we actually run 10 different ensembles for each domain: one for each leave-out fold. The ensemble components for the domains are given in Table 2. We winsorize the ensemble estimates at 0.001 and 0.999, which ensures we can calculate negative log likelihood of the final predictor.

Domain	**Ensemble Components**
Multiattribute Choice	Neural Network Distortion-Free Logit Relative Thinking L1 Complexity (2-parameters) L1 Complexity (3-parameters)
Intertemporal Choice	Neural Network Exponential Discounting Quasi-Hyperbolic Discounting Hyperbolic Discounting CPF Complexity
Lottery Choice	Neural Network Expected Utility Simplicity Theory Cumulative Prospect Theory Risk-neutral CDF Complexity Expected Utility CDF Complexity

Table 2: Ensemble component predictors.

H Restrictiveness Sampling Procedure

In our three domains, the admissable set of synthetic data $\mathcal{P}$ can be represented as a convex polytope $\mathcal{P} \equiv \{x \in \mathbb{R}^n : Ax \leq b\}$, where each dimension corresponds to the rate of choosing option a for a given choice problem, and where the linear program (A, b) encodes the Weak Dominance and Monotonicity constraints in addition to the constraints that each binary choice rate must lie in $[0, 1]$. We approximate uniform samples from $\mathcal{P}$ using the hit-and-run (HAR) sampler (Smith, 1984), a Markov-chain Monte Carlo algorithm designed to approximate uniform samples from a convex polytope (more generally, a convex set). Starting from an initial point x^0 in the interior of $\mathcal{P}$, HAR proceeds as follows:

1. Randomly sample a direction in $d^t \in \mathbb{R}^n$.
2. Uniformly sample along the intersection of the line $L = \{x^t + \theta d^t\}_{\theta \in \mathbb{R}}$ and $\mathcal{P}$ to obtain the next iterate x^{t+1}.

   That is, compute the bounds $[\theta_{min}, \theta_{max}]$ such that $x^t + \theta_{min} d^t$ and $x^t + \theta_{max} d^t$ lie on the boundary of the polytope, and draw $\theta \sim \mathcal{U}[\theta_{min}, \theta_{max}]$ and set $x^{t+1} = x^t + \theta d^t$.

   Since $\mathcal{P}$ is a convex polytope, $[\theta_{min}, \theta_{max}]$ can be computed as follows: letting $\lambda_i = \frac{(b-Ax^t)_i}{(Ad^t)_i}$, we have

$$\theta_{min} = \max_i \{\lambda_i : \lambda_i < 0\}$$
$$\theta_{max} = \min_i \{\lambda_i : \lambda_i > 0\}.$$

This defines a Markov chain with a stationary distribution equal to the desired uniform distribution over $\mathcal{P}$. Smith (1984); Lovász (1999) provide mixing time analysis for HAR.

To approximate a random sample of $K = 1000$ iid draws from the desired uniform distribution, we run the HAR sampler for 50,000 iterations using a thinning factor of M.[2] We then burn the first 10,000 iterations, and randomly sample K draws from the remaining 40,000 iterations. We use a thinning factor of $M = 1500$ for multiattribute and intertemporal choice and $M = 130000$ for lottery choice. The sampler was implemented using parallelized code developed in CUDA C.

As it is recommended that implementations of Markov chain Monte-Carlo samplers like HAR use a warm start for the initial value x^0 near the "center" of the polytope, i.e. away from "corners" (Lovász, 1999), we start the HAR sampler at the *p-center* of $\mathcal{P}$, a notion of the center of a polytope proposed by (Moretti, 2003). We use the iterative algorithm developed in Moretti (2003) to approximate the p-center, and initialize x^0 to this value.

[2]That is, we run HAR for $(M+1) \times 50{,}000$ iterations, storing every $(M+1)$th iteration. We use this thinning procedure due to computational memory constraints; given the high dimensionality of the polytope, it is infeasible to save every iteration of the sampler to memory.

I Experimental Interfaces

I.1 Multiattribute Binary Choice

Instructions 1/2

Please read these instructions carefully. There will be comprehension checks. If you fail these checks, you will be excluded from the study and you will not receive the completion payment.

In this study, you will make multiple decisions.

Your payment will consist of two components:

- Completion fee:
  If you pass all our **comprehension checks and complete the study**, you will receive a **completion fee of $5**.
- Additional bonus:
  In addition to the completion fee, you will have a chance to **earn a bonus based on one of your decisions**. You will face 50 decision screens over the course of this study. One of the decisions screens will be selected at random by the computer, and the option you selected on that screen will determine your bonus. **The average bonus is worth $6.50**. After your bonus is determined, the computer will run a lottery to determine if your bonus will actually be paid out to you. Your bonus will be paid out to you with probability 1/2.

Instructions 2/2

Choice task: which phone plan is a better deal?

On each decision screen, you will be presented with two cell-phone plans. Your task is to help **Amy**, a fictional customer, choose the lowest-cost plan. For the main part of the study, phone plans will consist of three components:

- Upfront cost of the device (charged annually)
- Recurring fee (paid in monthly installments)
- Data usage fee (charged per GB used)

The data usage fee is priced "per GB," and **Amy always uses 6 GB of data per month (72 GB annually)**. So, for a plan with a data usage fee of $1.00/GB, Amy would have to spend $6.00 per month on data, which amounts to $72.00 annually. Note: each plan offers the exact same services and devices; these plans *only* differ in their costs.

For the main part of the study, Amy's **annual phone budget** is **$700**. Your goal is to **guess which plan will leave Amy with the most money left over at the end of the year**. On each decision screen, you will be asked to make **two decisions:**

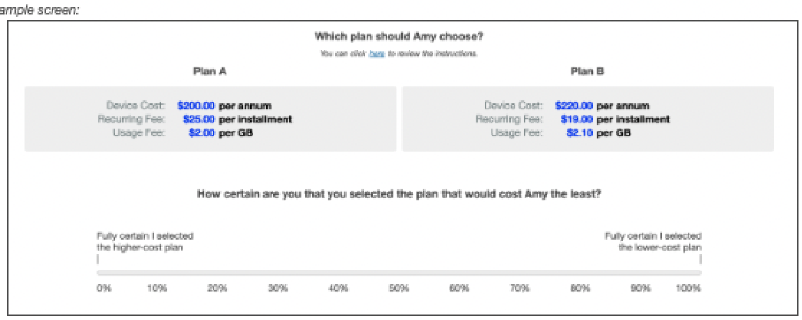
Step 1: Guess which phone plan has lower annual cost

- We will ask you to guess which plan will cost Amy less. You need to select exactly one plan.
- If this decision is randomly chosen for payment, your bonus will be 1 month's worth of Amy's total savings. This is equal to Amy's annual budget minus the cost of your chosen plan, divided by 12.
- This means that **to maximize your bonus, you should select the plan that you think will cost Amy the least over the year.**

Step 2: Indicate your certainty about your guess

- You may be uncertain over which plan actually has lower cost. Therefore, we will ask you to indicate how certain you are (in percent) that you've actually selected the lower-cost plan.
  - **For example, if you think it is 70% likely that you chose the lower-cost plan, you should set the slider to 70%.**
  - If you are certain that you chose the lower-cost plan, you should set the slider to 100%.

Example screen:



Factoring Amy's data usage, Plan A will cost her $644 over the year, whereas Plan B will cost $599, so Plan B is the lower cost plan.

Here is how your bonus would be determined in this example:
- If you selected Plan A, Amy would save $700 - $644 = $56, so your bonus would be $56 / 12 = $4.67.
- If you selected Plan B, Amy would save $700 - $599 = $101, so your bonus would be $101 / 12 = $8.42.

After your bonus is determined, the computer will randomly determine whether or not it will be paid out to you. You will actually receive your bonus 1/2 of the time.

Once you click the next button, the comprehension check questions will start!

Comprehension check

To verify your understanding of the instructions, please answer the comprehension questions below. If you get one or more of them wrong twice in a row, you will not be allowed to participate in the study and earn a completion payment. In each question, exactly one response option is correct.

You can review the instructions here.

1. To maximize your bonus, what should you select on each decision screen?

- I should try to select the phone plan bundle that I think would be the best deal for me personally.
- I should try to select the phone plan bundle that I think would be the best deal for most cell-phone users.
- I should try to select the phone plan bundle that will be best for Amy specifically.

2. How should you determine which bundle is best for Amy?

- The devices and services offered in each pair of bundles are identical, so I should try to select the bundle that will cost Amy the least given her data usage.
- I should usually select the bundle with a more expensive device, because that means Amy will get a nicer phone.
- I should always select a plan with a low per-GB data fee in case Amy's data usage increases.

3. Under the plan below, how much in **usage fees** would Amy pay **over the year**?

Device Cost:	**\$200.00 per annum**
Recurring Fee:	**\$25.00 per installment**
Usage Fee:	**\$1.00 per GB**

- \$1.00
- \$6.00
- \$12.00
- \$72.00

4. Suppose you chose plan A on a decision screen. Which of the following statements is correct?

- If I think it is 70% likely that plan A has the lowest cost, then I should set the certainty slider to 70%.
- If I think it is 70% likely that plan A has the lowest cost, then I should set the certainty slider to 100%.
- I should always set the certainty slider somewhere in the middle, even if I'm certain which plan has the lowest cost.
- I should always set the certainty slider to 100% even if I'm not certain which plan has the lowest cost.

Disclaimer

It is up to you how you wish to work on the tasks, but **we would prefer if you did not use a calculator** to help make your decisions.

You will need to complete **50 decision tasks** in total. You may take as much time for each task as you'd like, but please remember that the study was advertised for **30 minutes** and you will only be paid on that basis.

If you find that you don't have much time, you may look at the plans and make an informed guess about which one is lower cost. Again, it is up to you to how you wish to work on the tasks.

Which plan should Amy choose?

You can click here to review the instructions.

	Plan A	Plan B
Device Cost:	**$169.92 per annum**	**$169.92 per annum**
Recurring Fee:	**$10.26 per installment**	**$18.00 per installment**
Usage Fee:	**$3.49 per GB**	**$3.16 per GB**

How certain are you that you selected the plan that would cost Amy the least?

Fully certain I selected the higher-cost plan — Fully certain I selected the lower-cost plan

0% 10% 20% 30% 40% 50% 60% 70% 80% 90% 100%

I.2 Intertemporal Binary Choice

Instructions 1/2

Please read these instructions carefully. There will be comprehension checks. If you fail these checks, you will be excluded from the study and you will not receive the completion payment.

In this study, you will make multiple decisions.

Your payment will consist of two components:

- Completion fee:
  If you pass all our **comprehension checks and complete the study**, you will receive a **completion fee of $3.50**.
- Additional bonus:
  In addition to the completion fee, you will have a chance to **earn a bonus based on one of your decisions**. You will face 50 decision screens over the course of this study. With 1/5 chance, you will be selected to win a bonus payment. If this happens, one of the decision screens will be selected at random by the computer, and the option you selected on that screen will determine your **bonus**. The **maximal bonus** you can earn in this study is **$40**.

Instructions 2/2

Choice task: which payment option would you like to receive?

On each decision screen, you will be presented with two payment options. Each option will consist of payment amounts (in dollars), along with dates at which the payments are to be received.

On **each decision screen**, you will be **asked to indicate which payment option you prefer to receive.**

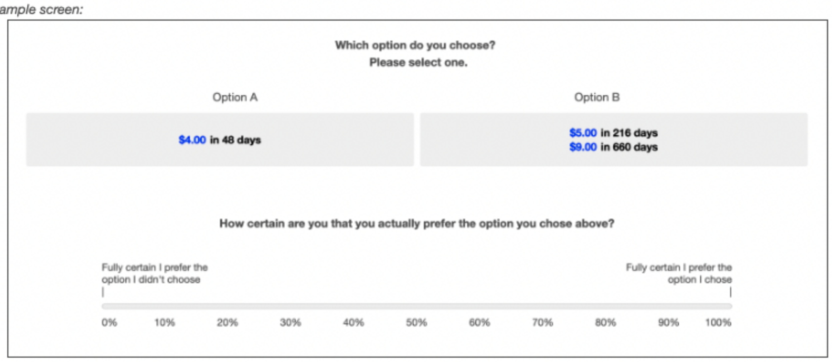
Step 1: Choose the payment option you prefer

- We will ask you to indicate which option you prefer to receive.
- If this decision is selected to determine your bonus, you will receive the payments in the option you chose, at the specified dates.

Step 2: Indicate your certainty about your choice

- You might feel uncertain about which payment option you actually prefer. Therefore, we will ask you to indicate how certain you are (in percent) that you actually prefer the option that you chose.
    - **For example, if you think it is 70% likely that you actually prefer the payment option that you chose, you should set the slider to 70%.**
    - If you are certain that you prefer the payment option you chose, you should set the slider to 100%.

Example screen:



If this decision is selected to determine your bonus:

- If you selected option A, you would receive a payment of $4.00 delivered to your account in 48 days.
- If you selected option B, you would receive a payment of $5.00 delivered to your account in 216 days and an additional payment of $9.00 delivered to your account in 660 days.

If a decision is selected to determine your bonus, the payments in the option you chose will be delivered to your account within 24 hours of the specified dates. When a payment is delivered, we will also send you a reminder through Prolific to cash out the payment.

Once you click the next button, the comprehension check questions will start!

Comprehension check

To verify your understanding of the instructions, please answer the comprehension questions below. If you get one or more of them wrong twice in a row, you will not be allowed to participate in the study and earn a completion payment. In each question, exactly one response option is correct.

You can review the instructions here.

1. How is your bonus determined?

I will make multiple decisions, and every one of them will get paid. Thus, I can strategize across decisions.

I will make multiple decisions. The computer will randomly select one of them, and my potential bonus will depend on my decision in this one question. Thus, there is no point for me in strategizing across decisions.

2. Suppose that you chose the following payment option in one of the decisions.

$5.00 in 216 days
$9.00 in 660 days

Which of the following statements is correct?

If this decision is selected for payment, I will receive $14 within 24 hours.

If this decision is selected for payment, I will receive $14 in total: $5 in 216 days, and $9 in 660 days.

3. Please select the statement that is true.

If I think it is 70% likely that I actually prefer Option A, then I should set the slider to 100%.

If I think it is 70% likely that I actually prefer Option A, then I should set the certainty slider to 70%.

4. Which of the following statements is correct?

My bonus will be based completely on which option I choose in Step 1, regardless of how much uncertainty I express in Step 2.

If I indicate that I am uncertain about my choice, then the bonus I receive will be a combination of Options A and B.

Which option do you choose?
Please select one.

Option A	Option B
$0.50 in 24 days **$19.50 in 360 days**	**$14.00 in 144 days** **$3.50 in 360 days**

J Additional Results and Applications

J.1 Characterization of Generalized L_1-Complexity

Consider the generalized L_1-complexity representation introduced in Definition 8, where given a partition P over features $I = \{1, ..., n\}$, choice rates are given by

$$\rho(x, y) = G\left(\frac{U(x) - U(y)}{d_{L1}(x, y)}\right)$$

for continuous, strictly increasing G, where $U(x) = \sum_{E \in P} u_E(x_E)$ and $d_{L1}(x, y) = \sum_{E \in P} |u_E(x_E) - u_E(y_E)|$ for continuous, non-trivial $u_E : X_E \to \mathbb{R}$.

Notice that in Definition 8, it is without loss to assume that utility is additively separable over each feature: if utility is not additively separable over a set of features $B \subseteq I$, the analyst can simply combine those features as a single joint feature $X_B = \times_{i \in B} X_i$, and re-define the representation over the set of joint features. We will therefore provide axiomatic foundations for the generalized L_1-complexity representation $(G, P, (u_E)_{E \in P})$ where each u_E is additively separable on its domain.

Our axiomatization will involve the same conditions as in Theorem 3, the characterization result for additively-separable L_1 complexity, aside from a slight weakening of M4 (Dominance).

To state the new condition, we introduce a behavioral notion of resolvability, which captures whether or not the DM understands how to resolve tradeoffs between a subset of features. Say that $E \subseteq I$ is *resolvable* if for any $x, x', y \in X$ where $\rho(x'_E x, x) = 1/2$, we have $\rho(x'_E x, y) = \rho(x, y)$. In words, if the set of features E is resolvable, choice between two options (x, y) is unaffected by the nature of tradeoffs present among features in E, so long as the total value each option delivers across those features is unchanged. Additionally, say that E is *maximally resolvable* if there does not exist $E' \supset E$ such that E' is resolvable, and say that E is *non-resolvable* if for all $i, j \in E$, $\{i, j\}$ is not resolvable.

M4*. **Dominance*:** If $x >_D y$, then $\rho(x, y) \geq \rho(w, z)$ for all $w, z \in X$, where the inequality is strict if $\rho(w_E z, z) < 1/2$ for some maximally resolvable $E \subseteq I$.

Notice that M4* weakens M4; under M4*, the inequality $\rho(x, y) \geq \rho(w, z)$ is strict only if z is undominated by w over all maximally resolvable collections of features, as opposed to over all features as in M4.

Theorem 6. *Suppose that all features are non-null and that there are at least three non-resolvable features. Then a binary choice rule ρ satisfies M1, M3, M4*, M5, M7–M8 if and only if it has a generalized L_1 complexity representation $(G, P, (u_E)_{E \in P})$, where each u_E is additively separable.*

Proof. The proof of necessity of M1, M3, M4*, M5, M7–M8 is straightforward, so we focus on sufficiency. Let $\succeq$ be the stochastic preference relation induced by ρ. $\succeq$ satisfies coordinate independence and inherits continuity from ρ, and since we have at least 3 non-null attributes, we invoke Debreu (1983) to conclude that $\succeq$ has an additively separable representation: there exists $u_i : X_i \to \mathbb{R}$, continuous, such that

$$x \succeq y \iff \sum_k u_k(x_k) \geq \sum_k u_k(y_k)$$

Since all attributes are non-null and the X_k are connected, each $u_k(X_k)$ is a non-trivial interval of $\mathbb{R}$. Since the representation is unique up to cardinal transformations, we can without loss assume that for each $k \in I$, $u_k(X_k)$ contains 0, and furthermore, since $u_k(X_k)$ is a non-trivial interval, that $u_k(X_k)$ contains a non-trivial open interval around 0. For all $k \in I$, let $\overline{u}_k = \sup u_k(X_k)$ and $\underline{u}_k = \inf u_k(X_k)$, taken with respect to the extended real line.

For all $x \in X$, define $\tilde{x} = (u_1(x_1), \ldots, u_k(x_k)) \in \mathbb{R}^n$. Let $\tilde{X} = \{\tilde{x} \in \mathbb{R}^n : x \in X\}$. Let $\tilde{\mathcal{D}} = \{(a,b) \in \tilde{X} : a \neq b\}$ and define $\phi : \tilde{\mathcal{D}} \to \mathcal{D}$ satisfying $\phi(a,b) \in \{(x,y) \in \mathcal{D} : \tilde{x} = a, \tilde{y} = b\}$, and define $\tilde{\rho} : \tilde{\mathcal{D}} \to [0,1]$ by $\tilde{\rho}(a,b) = \rho(\phi(a,b))$. Lemma 8 implies that $\tilde{\rho}$ is a binary choice rule on $\tilde{\mathcal{D}}$ and does not depend on the selection made by ϕ: in particular, we have $\tilde{\rho}(\tilde{x},\tilde{y}) = \rho(x,y)$ for all $(x,y) \in \mathcal{D}$. This in turn implies that $\tilde{\rho}$ inherits our axioms M1, M3, M4*, M5, M7–M9.

For all $E \subseteq \mathcal{P}$, let $u_E : X_E \to \mathbb{R}$ be defined by $u_E(x_E) = \sum_{k \in E} u_k(x_k)$ for all $x \in X$. Note that if there exists a strictly increasing, continuous function G and a partition P of I such that

$$\tilde{\rho}(a,b) = G\left(\frac{\sum_{E \in P}(\sum_{k \in E} a_k - \sum_{k \in E} b_k)}{\sum_{E \in P} |\sum_{k \in E} a_k - \sum_{k \in E} b_k|}\right)$$

for all $(a,b) \in \tilde{\mathcal{D}}$ such that $\sum_{E \in P} |\sum_{k \in E} a_k - \sum_{k \in E} b_k)| > 0$, we are done, as this implies that for any $(x,y) \in \mathcal{D}$ such that $\sum_{E \in P} |u_E(x_E) - u_E(y_E)| > 0 \iff \sum_{E \in P} |\sum_{k \in E}(\tilde{x}_k - \tilde{y}_k)| > 0$, we have

$$\rho(x,y) = \rho(\tilde{x},\tilde{y}) = G\left(\frac{\sum_{E \in P}(u_E(x_E) - u_E(y_E))}{\sum_{E \in P} |u_E(x_E) - u_E(y_E)|}\right)$$

and by construction, for any $(x,y) \in \mathcal{D}$ such that $\sum_{E \in P} |u_E(x_E) - u_E(y_E)| = 0$, we have $x \sim y$ and so $\rho(x,y) = 1/2$.

In what follows, we will work with $\tilde{\rho}$ defined on $\tilde{X}$ and drop the $\sim$ in our notation. Following Lemmas 9–13 in the proof of Theorem 3, Continuity, Moderate Transitivity, Separability, and Tradeoff Congruence imply that ρ satisfies scale and translation invariance and therefore satisfies Linearity; again following the same construction as in the proof of Theorem 3, linearly extend ρ to $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$. Say that $i =_R j$ whenever $\{i,j\}$ is resolvable.

Lemma 19. *If $i =_R j$ and $j =_R k$, then $i =_R k$.*

Proof. Suppose $i =_R j$ and $j =_R k$. Fix $x, x', y \in \mathbb{R}^n$ such that $\rho(x'_{\{i,k\}}x, x) = 1/2$. We want to show that $\rho(x'_{i,k}x, y) = \rho(x, y)$. Define $z, z' \in \mathbb{R}^n$ satisfying

$$z_l = \begin{cases} x'_i & l = i \\ x_i + x_j - x'_i & l = j \\ x_k & l = k \end{cases} \qquad w_l = \begin{cases} x'_i & l = i \\ x_j & l = j \\ x'_k & l = k \end{cases}$$

Notice that $z_i + z_j = x_i + x_j$ implies that $\rho(z_{\{i,j\}}x, x) = 1/2$. Notice also that

$$\begin{aligned} z_j + z_k &= x_i + x_j + x_k - x'_i \\ &= x_j + x'_k \\ &= w_j + w_k \end{aligned}$$

and so $\rho(w_{\{j,k\}}z, z) = 1/2$, where the second equality follows from the fact that $\rho(x'_{\{i,k\}}x, x) = 1/2 \implies x_i + x_k = x'_i + x'_k$. We have

$$\begin{aligned} \rho(x, y) &= \rho\left(z_{\{i,j\}}x, y\right) && \text{since } i =_R j \\ &= \rho\left(z_{\{j,k\}}\left(x'_{\{i\}}x\right), y\right) \\ &= \rho\left(w_{\{j,k\}}\left(x'_{\{i\}}x\right), y\right) && \text{since } j =_R k \\ &= \rho\left(w_{\{i,k\}}x, y\right) \\ &= \rho\left(x'_{\{i,k\}}x, y\right) \end{aligned}$$

as desired.

□

Lemma 19 implies that $=_R$ defines an equivalence relation on I, and so the equivalence classes of $=_R$ form a partition on I. Denote this partition by P. Note that since by hypothesis there exists a set of 3 non-resolvable features, we have $|P| > 3$. Furthermore, it can be easily shown that each $E \in P$ is resolvable, since all pairs of attributes in E are resolvable.

Let $\hat{X} = \times_{E \in P}\mathbb{R}$, and let $\hat{\mathcal{D}} = \{(a, b) \in \hat{X} \times \hat{X} : a \neq b\}$. For $a \in \hat{X}$, we will abuse notation by letting $a_E \in \mathbb{R}$ denote the dimension of a corresponding to the partition element $E \in P$. For all $x \in \mathbb{R}^n$, define $\hat{x} \in \hat{X}$ where $\hat{x}_E = \sum_{k \in E} x_k$ for all $E \in P$. Define $\psi : \hat{X} \to \mathbb{R}^n$ by

$$\psi(a)_i = \begin{cases} a_E & i = \min E \text{ for some } E \in P \\ 0 & \text{otherwise} \end{cases}$$

for all $a \in \hat{X}$, and define $\hat{\rho}$ on $\hat{X}$ by $\hat{\rho}(a,b) = \rho(\psi(a), \psi(b))$ for all $(a,b) \in \hat{\mathcal{D}}$. Note that $\hat{\rho}$ is a binary choice rule, and directly inherits M1, M2, M3, and M5 from ρ.

To see that $\hat{\rho}$ satisfies M4 (Dominance), fix $(a,b) \in \mathcal{D}$ satisfying $\hat{\rho}(a_E b, b) \geq 1/2$ for all $E \in P$, with a strict inequality for at least one $E \in P$, and fix any $c, d \in \mathcal{D}$. This implies $\rho(\psi(a)_{\{k\}}\psi(b), \psi(b)) \geq 1/2$ for all $k \in I$ with a strict inequality for at least one k, and since ρ satisfies M4*, we have $\rho(\psi(a), \psi(b)) \geq \rho(\psi(c), \psi(d)) \implies \hat{\rho}(a,b) \geq \hat{\rho}(c,d)$. Furthermore, suppose that $\hat{\rho}(c_E d, d) < 1/2$ for some $E \in P$. This implies that $\rho(\psi(c)_E \psi(d), \psi(d)) < 1/2$, and so again invoking M4* of ρ we have $\rho(\psi(a), \psi(b)) > \rho(\psi(c), \psi(d)) \implies \hat{\rho}(a,b) > \hat{\rho}(c,d)$; $\hat{\rho}$ therefore satisfies M4.

By construction, $\hat{X}$ contains at least three attributes. Note also that attributes of $\hat{X}$ are non-null with respect to $\hat{\rho}$, a property inherited from ρ. By Theorem 1, there exists G continuous, strictly increasing such that for all $(a,b) \in \hat{\mathcal{D}}$

$$\hat{\rho}(a,b) = G\left(\frac{\sum_{E \in P}(a_E - b_E)}{\sum_{E \in P}|a_E - b_E|}\right).$$

Now, fix any $(x,y) \in \mathcal{D}$ such that $\sum_{E \in P} |\sum_{k \in E} x_k - \sum_{k \in E} y_k| > 0$. We have

$$\begin{aligned}
\rho(x,y) &= \rho(\psi(\hat{x}), \psi(\hat{y})) && \text{since each } E \in P \text{ is resolvable} \\
&= \hat{\rho}(\hat{x}, \hat{y}) \\
&= G\left(\frac{\sum_{E \in P}(\hat{x}_E - \hat{y}_E)}{\sum_{E \in P}|\hat{x}_E - \hat{y}_E|}\right) \\
&= G\left(\frac{\sum_{E \in P}(\sum_{k \in E} x_k - \sum_{k \in E} y_k)}{\sum_{E \in P}|\sum_{k \in E} x_k - \sum_{k \in E} y_k|}\right) && \text{by construction}
\end{aligned}$$

as desired. □

J.2 Joint vs. Separate Evaluation

Consider the finding of "scope insensitivity" in contingent valuation tasks: when assigning a monetary valuation to a policy with a quantifiable impact, individuals are insufficiently sensitive to the impact of the policy (see Toma and Bell, 2024, for a review).

To take an example, consider the following environmental policies:

$$\text{Program } x : \text{save 5400 endangered birds}$$
$$\text{Program } y : \text{save 12000 endangered birds.}$$

Frederick and Fischoff (1998) find that when asked to assign a dollar value to one of the two policies in a between-subjects design, respondents' valuations are highly insensitive to the impact of the policy, i.e. the number of birds saved. While this insensitivity could simply reflect respondents' preferences, Frederick and Fischoff (1998) also find that respondents' valuations are far more sensitive to policy impact when they are asked to value multiple policies jointly in a within-subjects design, as opposed to valuing a single policy in the between-subjects design. That is, individuals' valuations of options appear more sensitive to fundamentals when made jointly as opposed to separately. Subsequent work has found similar joint vs. separate evaluation effects, both in policy impact evaluation (e.g. Toma and Bell, 2024) and more generally (see Hsee et al., 2009, for a review).

Our model provides a natural explanation of these findings: x and y are ostensibly difficult to compare to monetary values due to the tradeoffs involved, but are easy to compare to each other: all else equal, it is clearly better to save more birds than less. As a result, the compression effects in our model cause x and y to be valued too similarly when the options are valued separately, but the additional information that y is superior to x contrasts the valuations of x and y away from each other when the options are valued jointly, causing the DM to appear more sensitive to policy impact.

We extend the multiple price list valuation framework presented in the main text — which considers independent valuations of a single good — to joint valuations as follows. Given options x, y to be valued jointly against a price list $Z = (z^1, ..., z^n)$, define a *joint valuation task* (x, y, Z) as the binary menu sequence $A^{x,1}, ..., A^{x,n}, A^{y,1}, ..., A^{y,n}$, where $A^{x,1}, ..., A^{x,n} = \{x, z^1\}, ..., \{x, z^n\}$ and $A^{y,1}, ..., A^{y,n} = \{y, z^1\}, ..., \{y, z^n\}$.[3] That is, the DM values both x and y against the price list Z, where both options x and y are contained in the choice context.[4]

Notice that restricting to either price list $A^{x,1}, ..., A^{x,n}$ or $A^{y,1}, ..., A^{y,n}$, this procedure yields a single switching point in the DM's choices: for any signal realization, there is an index $R_x \in \{1, ..., n, n+1\}$ for which the DM chooses $x \in A^{x,k}$ for all $k \geq R_x$ and the price $z^k \in A^{x,k}$ for all $k < R_x$; let R_y be defined analogously. Let $R(x, Z|y)$ and $R(y, Z|x)$ denote the distribution of the switching points R_x and R_y induced by the DM's choice probabilities. We

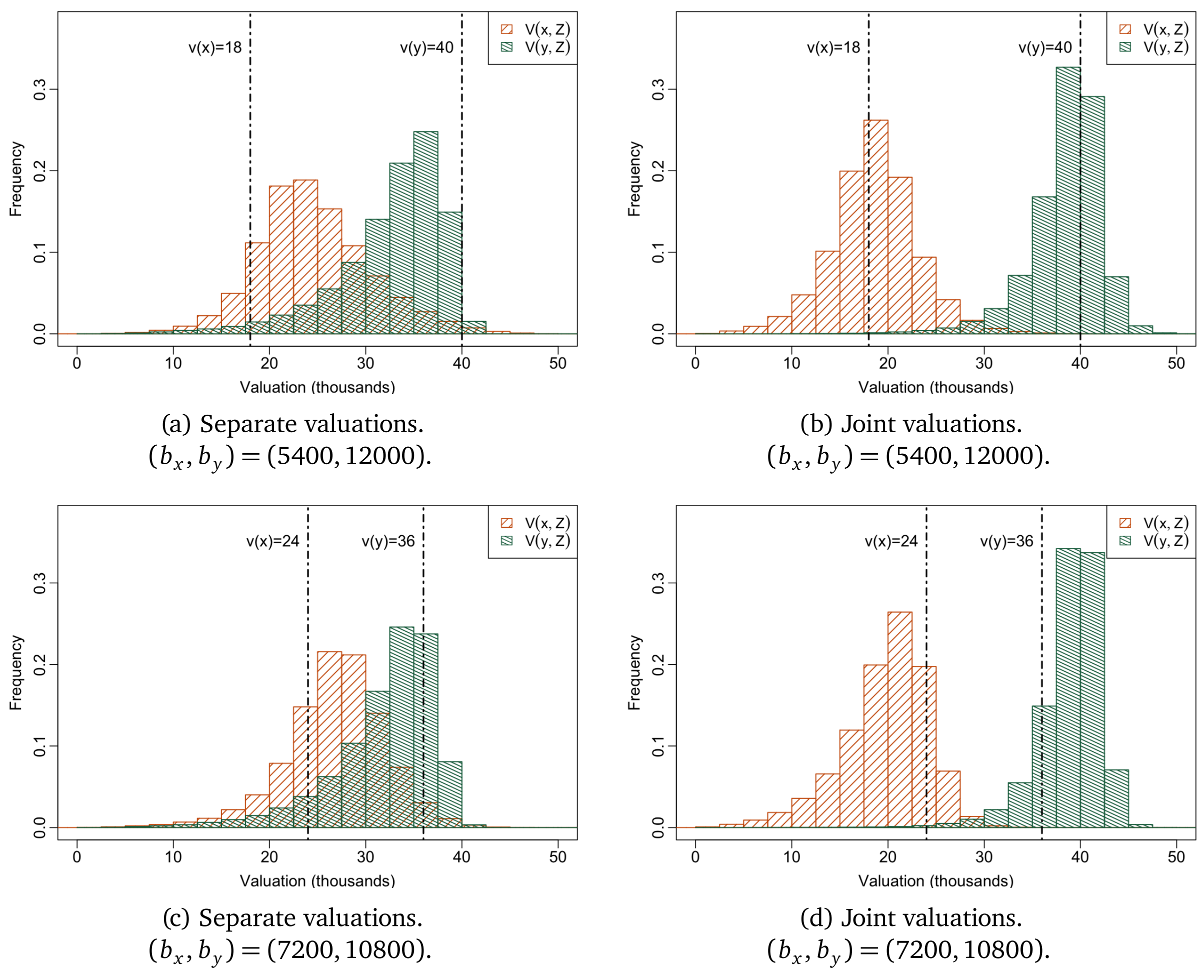
[3] As in the main text, we assume the options in the price list Z are unambigiously ranked, i.e. $\tau_{z^i z^j} = \infty$ for all i, j, and that v_{z^k} is strictly increasing in k.

[4] This setting can be straightforwardly extended to model joint valuations of three or more choice options.

will be interested in how the DM's joint valuations $R(x, Z|y)$ and $R(y, Z|x)$ compare to the separate valuations $R(x, Z)$ and $R(y, Z)$.

To adapt the example to our setting, suppose that each outcome $w = (w_1, w_2)$ is described by two attributes, monetary payments w_1 and number of birds saved w_2, where $U(w) = w_1 + 10/3 \cdot w_2$ – that is, the DM values the life of each bird at \$3.33. The ease of comparison τ has an L_1-complexity representation $\tau^{L1}_{xy} = H\left(\frac{|U(x)-U(y)|}{d_{L1}(x,y)}\right)$ for which $H(1) = \infty$.

The DM is tasked with valuing the policies $x = (0, b_x)$, $y = (0, b_y)$, where b_x and b_y denote the number of birds saved by the two policies, against a price list $Z = (z^1, ..., z^n)$ of monetary amounts, where each $z^k = (m_k, 0)$. We consider two settings: one where each policy is valued separately, and one where policies are valued jointly. As in the applications to price list valuations in the main text, we associate each switching point R with a valua-



(a) Separate valuations. $(b_x, b_y) = (5400, 12000)$.

(b) Joint valuations. $(b_x, b_y) = (5400, 12000)$.

(c) Separate valuations. $(b_x, b_y) = (7200, 10800)$.

(d) Joint valuations. $(b_x, b_y) = (7200, 10800)$.

Figure 1: Separate and joint valuations of $x = (0, b_x)$, $y = (0, b_y)$ against Z. Z contains evenly spaced payments ranging from 0 to 50 with $|Z| = 21$. τ has a L_1-complexity representation with $\beta = (1, 10/3)$ and G given by (1) with $\kappa = 0, \gamma = 0.5$. Priors are distributed $Q \sim U[0, 1]$.

tion equal to $1/2[w_{R-1} + w_R]$; each distribution over switching points therefore induces a distribution over valuations.

Figure 1 plots these distributions simulated from our model under separate vs. joint valuation, using two sets of (b_x, b_y) values; here the vertical dashed lines indicate the true valuations of x and y. Focusing first on Figures 1a and 1c, notice that compression effects in our model produce scope insensitivity when x and y are valued separately; relative to their true valuations, the DM appears insensitive to variation in policy impact. When the same options are valued jointly as in Figures 1b and 1d, however, the valuation distributions are repulsed away from each other and so the DM appears more sensitive to impact.

Importantly, our model does not predict that joint valuation necessarily improves the accuracy of the DM's assessments. While Figure 1b shows that joint evaluation improves valuations in the case where the difference between b_x and b_y is large, notice that when the difference between b_x and b_y is relatively small, joint evaluation causes the DM to *overstate* the true difference between the impacts of the policies, as Figure 1d highlights.

These results also relate to "coherent arbitrariness" (Ariely et al., 2003): the idea that whereas agents' elicited preferences are unstable and subject to variant to irrelevant changes in context, they nevertheless adhere to coherent comparative statics. In the joint valuations in our model, even though the DM's valuations are noisy and in some cases systematically biased, they cohere with dominance.

J.3 Front-End Delays and Compounding

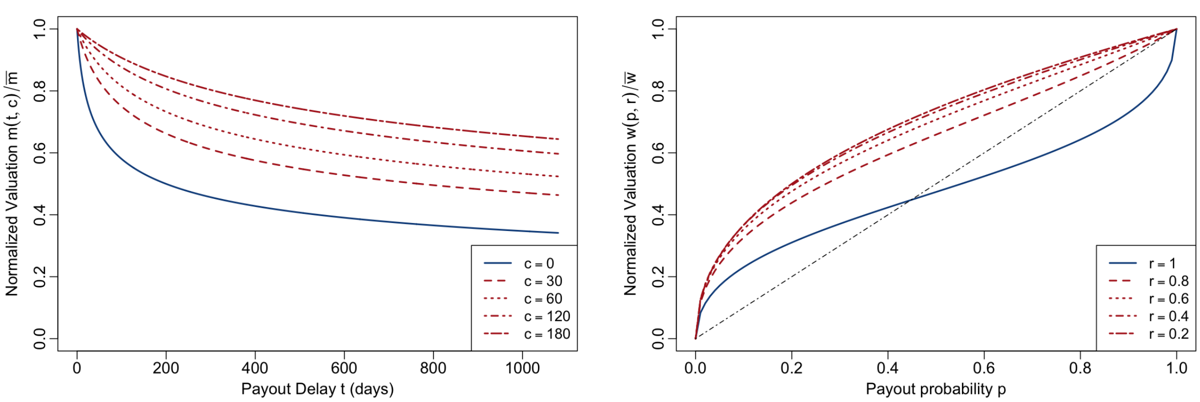
Invariance to Front-End Delays. Consider the present value equivalents task analyzed in Section 3.4 of the main text, where the DM values delayed payments $v = (\overline{m}, t_v)$ in terms of a price list $Z = (z^1, , ..., z^n)$ of immediate payments $z^k = (m_k, 0)$. In addition to the finding of apparent hyperbolic discounting in these tasks, experimental work has shown that adding a front-end delay to the valuation task has a relatively small impact on the required rate of return implied by subjects' valuations (see Cohen et al., 2020, for a review). That is, subjects' valuations exhibit near-stationarity with respect to front-end delays, which is seemingly at odds with the high degree of apparent hyperbolicity implied by their valuations.

To see this, consider the predictions of a noise-free model of discounted utility (DU), where the value of a delayed payment (m, t) is given by $u(m)d(t)$ for some strictly decreasing, potentially non-stationary discount function d satisfying $d(0) = 1$. Under such a model, the DM's valuation of v for a given front-end delay c is given by

$$m^*(t,c) = u^{-1}\left(\frac{d(t+c)}{d(c)}u(\overline{m})\right)$$

This implies that if the DM's discount function is hyperbolic or otherwise non-stationary, valuations under DU must be sensitive to the front-end delay c. In particular, assuming u strictly increasing, $m^*(t,c)$ is constant in c if and only if d is stationary, i.e. an exponential discount function. Figure 2a illustrates: if we calibrate a hyperbolic discount function $d(t)$ to the present value equivalents data from our valuation experiment[5] (solid blue curve), DU predicts a marked out-of-sample decrease in the required rate of return when front-end delays are incorporated, i.e. when $c > 0$ (red dashed/dotted curves). In contrast, the literature consistently documents pronounced hyperbolic discounting, yet minimal effects of front-end delays Cohen et al. (2020).

Our model can rationalize the simultaneous existence of hyperbolicity in present value equivalents as well as the weak effect of front-end delays. Intuitively, our model generates apparent hyperbolic discounting in valuations through pull-to-center effects that result from the difficulty of trading off money and delays, a difficulty that is present regardless of whether the valuation task features front-end delays. In particular, the ease of comparison under CPF-complexity satisfies a stationarity property (see Appendix B.1), generating the *same* degree of apparent hyperbolicity in valuations irrespective of front-end delays.



(a) Normalized valuations $m^*(p,r)/\overline{w}$ in present value equivalents task with front-end delays under Hyperbolic Discounting. Assumes linear u and $d(t) = (1+\iota t)^{-\zeta/\iota}$, for $\iota = 0.095$, $\zeta = 0.022$.

(b) Normalized valuations $w^*(p,r)/\overline{w}$ of compounded certainty equivalents task under CPT. Assumes linear u and $\pi(p) = \frac{\chi p^{\nu}}{\chi p^{\nu}+(1-p)^{\nu}}$, for $\nu = 0.5$, $\chi = 0.9$.

Figure 2: Predictions of deterministic models of non-stationary discounting and non-linear probability weighting under front-end delays and compounding.

[5]See Section 4.3 from the main text.

Formally, adding a front-end delay $c > 0$ amounts to a valuation task $(v(c), Z(c))$, wherein the DM values $v(c) \equiv (\overline{m}, t_v + c)$ against $Z(c) \equiv (z^1(c),, ..., z^n(c))$ consisting of delayed payments $z^k(c) \equiv (m_k, c)$. Let $PVE_c(v, Z) = 1/2[m_{R(v(c),Z(c))-1} + m_{R(v(c),Z(c))}]$ denote the distribution over the DM's valuations of v given a front-end delay c; notice that $PVE_0(v, Z)$ corresponds to valuations in a standard present value equivalents task with no front-end delay, as defined in Section 3.4.

Under the maintained assumption that τ has a CPF-complexity representation $\tau_{xy} = H\left(\frac{|PV(x)-PV(y)|}{d_{CPF}(x,y)}\right)$, the ease of comparison between the delayed payment $v(c)$ and each price $z^k(c)$ is invariant to the front-end delay c. As a result, the model generates the prediction that $PVE_0(v, Z) = PVE_c(v, Z)$ — i.e. the DM's valuations are stationary with respect to front-end delays. At the same time, however, pull-to-center effects in the model can generate apparent hyperbolicity in $PVE_0(v, Z)$, as demonstrated in Section 3.4.[6]

Invariance to Compounding. Our model predicts an analogous invariance to compounding in certainty equivalent tasks. Recall the certainty equivalents task analyzed in Section 3.4, where the lottery $l = (\overline{w}, p)$ is valued in terms of certain payments $Z = (z^1, ..., z^n)$ with $z^k = (w_k, 1)$. Now consider a *compounded* task, where the payout probabilities of all options is compounded by a probability $r \leq 1$ – that is, the DM values $l(r) \equiv (\overline{w}, rp)$ in terms of the compounded certain payments $Z(r) \equiv (z^1(r), ..., z^n(r))$, where $z^k(r) \equiv (w_k, r)$.

Consider the predictions of Cumulative Prospect Theory (CPT), in which the value of a simple lottery (w, p) is given by $\pi(p)u(w)$ for some weighting function $\pi : [0, 1] \to [0, 1]$ satisfying $\pi(0) = 0$, $\pi(1) = 1$. Under CPT, the DM's valuation of $l(r)$ for a given compounding factor r is given by

$$w^*(p, r) \equiv u^{-1}\left(\frac{\pi(pr)}{\pi(r)}u(\overline{w})\right).$$

This expression implies if the DM's probability weighting function is non-linear, valuations will be sensitive to the compounding factor r; in other words, assuming u strictly increasing, $w^*(p, r)$ is constant in r if and only if π is linear, i.e. the DM has expected utility preferences. Figure 2b illustrates: if we calibrate the CPT weighting function $\pi(p)$ to the certainty equivalents data from our valuation experiment[7] (solid blue curve), CPT predicts a markedly

[6]Front-end delay experiments do tend to find some difference in required rates of return (Cohen et al., 2020), consistent with the idea that individuals have some degree of "true" present-biased discounting. Under the generalized CPF-complexity measure (Definition 8), which allows for a non-stationary discount function, the model can rationalize these differences.

[7]See Section 4.3 from the main text.

different out-of-sample pattern in the DM's valuations in the compounded task, i.e. when $r < 1$ (red dashed/dotted curves).

In contrast to these predictions, McGranaghan et al. (2024) find that normalized valuations exhibit a virtually *identical* inverse-S pattern in the compounded task as in the standard certainty equivalents task. As they note, their findings are inconsistent with models of probability weighting in the Kahneman and Tversky (1979) tradition of prospect theory.

Our model can rationalize the invariance of inverse-S weighting to compounding just as it rationalizes the invariance of hyperbolicity to front-end delays. Intuitively, the apparent probability weighting in our model is a consequence of pull-to-center distortions caused by the difficulty of trading off probabilities and payouts in certainty equivalent tasks, a difficulty that remains even when the prospects in the task are compounded. Formally, under the maintained assumption that τ has a CDF-complexity representation $\tau_{xy} = H\left(\frac{|EU(x)-EU(y)|}{d_{CDF}(x,y)}\right)$, the ease of comparison between between $l(r)$ and each $z^k(r)$ in the price list is invariant to r, which implies that the DM's valuations are invariant to compounding.[8] At the same time, however, pull-to center effects in our model can generate apparent inverse-S probability weighting, as Section 3.4 illustrates. Our model can therefore rationalize the simultaneous presence of apparent non-linear probability-weighting in certainty equivalents, as well its invariance to compounding.

J.4 Uncertainty Equivalents

Call $l = (p; w, w')$ a binary lottery if it pays out w with probability q and w' otherwise. Consider the "uncertainty equivalents" valuation paradigm studied in Andreoni and Sprenger (2011): a binary lottery $l_p = (p; w_0, w_1)$ for $0 < w_0 < w_1$ is valued in terms of probability equivalents: the probability q that makes the binary lottery $(q; w_1, 0)$ indifferent to l_p. Note that expected utility theory predicts that q should be linear in p, whereas a number of models of non-EU risk preferences (e.g. probability weighting) predict a non-linear relationship.

Andreoni and Sprenger (2011) elicit probability equivalents q for a range of $p \in [0, 1]$, and find an essentially linear relationship between q and p along nearly the full range of p, which they show is inconsistent with models of non-linear probability weighting and disappointment aversion (Bell, 1985; Loomes and Sugden, 1986; Gul, 1991), both of which predict a non-linear relationship. In other words, the probability weighting observed in classic valuation tasks is seemingly at odds with behavior in uncertainty equivalent tasks.

[8]As Appendix B.1 shows, CDF-complexity satisfies a stochastic analog of the independence axiom for binary comparisons.

What does our model predict in this situation? Formally, the DM values l_p against a probability list $Z = (z^1, ..., z^n)$, where each z^k is a binary lottery $z^k = (q_k, 0, w_1)$. We will restrict attention to probability lists that are *adapted* in the same sense as in the main text: those consisting of evenly-spaced probabilities that extend to natural dominance points (i.e. $q_1 = 1$, $q_n = 1-p$, and $q_k - q_{k-1}$ is constant in k). As before, we define the probability equivalent associated with a switching point $R(l_p, Z)$ to be $PE(l_p, Z) = 1/2[q_{R(l_p,Z)-1} + q_{R(l_p,Z)}]$.

Notice a key property of the uncertainty equivalents paradigm: the true valuation lies in the same relative position along the range of undominated probabilities in the price list $[1-p, 1]$, regardless of p. To see this, fix any Bernoulli utility function u. The DM's true uncertainty equivalent equals $q^*(l_p) = (1-p) + \frac{u(w_0)}{u(w_1)}p$, and so the relative position of $q^*(l_p)$ along $[1-p, 1]$ is given $\frac{q^*(l_p)-(1-p)}{1-(1-p)} = \frac{u(w_0)}{u(w_1)}$, which is constant to p. This implies that relative position of the DM's uncertainty equivalents along $[1-p, 1]$, while distorted by pull-to-center effects, will also be constant in p. Our model therefore predicts that the uncertainty equivalents of l_p will be linear in p. The following result formalizes this intuition.

Proposition 13. *Suppose Z is adapted to l_p and $|Z|$ is fixed at n. If τ has a CDF-complexity representation, then $\mathbb{E}[PE(l_p, Z)]$ is linear in p.*

Proof. Fix any p. Since Z is adapted to l_p, the payoff probabilities of the binary lotteries in the price list $z^k = (q_k; w_1, 0)$ are given by $q_k = \lambda_k p + (1-p)$ where $\lambda_k = \frac{n-k}{n-1}$. Notice that

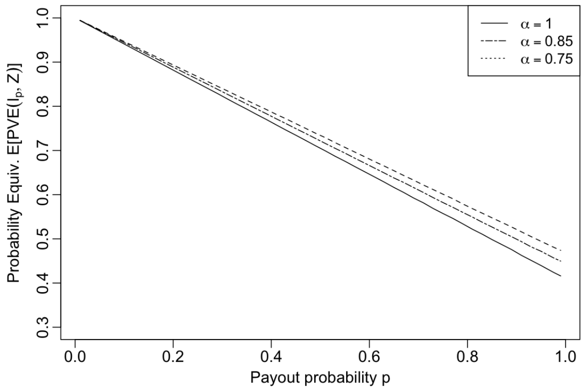
$$\begin{aligned}\tau_{l_p,z^k} &= \frac{|(1-\lambda_k)p(u(0)-u(w_0)) + \lambda_k p(u(w_1)-u(w_0))|}{(1-\lambda_k)p|u(0)-u(w_0)| + \lambda_k p|u(w_1)-u(w_0)|} \\ &= \frac{|(1-\lambda_k)(u(0)-u(w_0)) + \lambda_k u(w_1)-u(w_0))|}{(1-\lambda_k)|u(0)-u(w_0)| + \lambda_k|u(w_1)-u(w_0)|}\end{aligned}$$

and so for each k, τ_{l_p,z^k} is constant in p. This implies that the distribution of switching points $R(l_p, Z)$ is constant in p. This in turn implies that

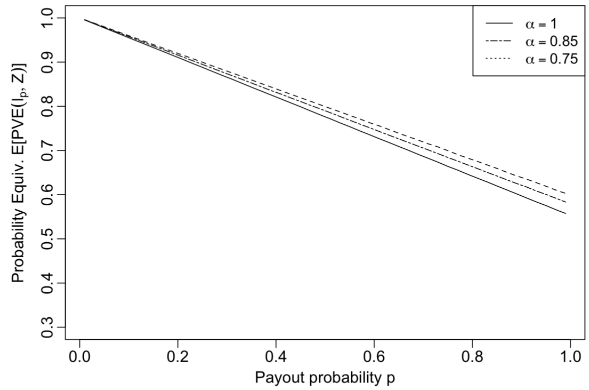
$$\begin{aligned}\mathbb{E}[PE(l_p, Z)] &= 1/2 \cdot \mathbb{E}\left[q_{R(l_p,Z)-1} + q_{R(l_p,Z)}\right] \\ &= 1 - \left[1 - 1/2 \cdot \mathbb{E}\left(\lambda_{R(l_p,Z)-1} + \lambda_{R(l_p,Z)}\right)\right]p\end{aligned}$$

and so $\mathbb{E}[PE(l_p, Z)]$ is linear in p. □

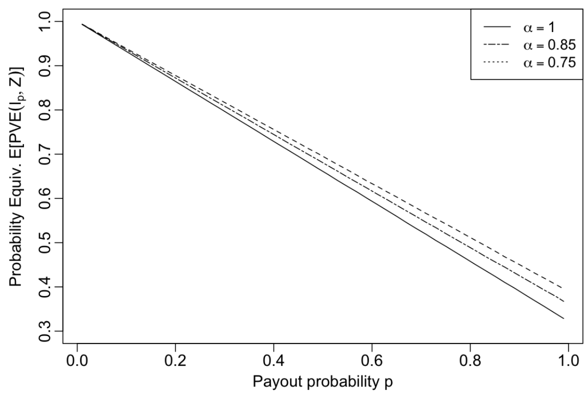
Figure 3 plots the simulated probability equivalents $\mathbb{E}[PE(l_p, Z)]$ as a function of p for the values of (w_0, w_1) considered in Andreoni and Sprenger (2011). The simulated probability equivalents indeed exhibit a linear relationship with p, largely consistent with the results of Andreoni and Sprenger (2011). As such, our model is able to simultaneously explain both



(a) $(w_0, w_1) = (\$10, \$30)$.



(b) $(w_0, w_1) = (\$30, \$50)$.



(c) $(w_0, w_1) = (\$10, \$50)$.

Figure 3: Probability equivalents $\mathbb{E}[PE(l_p, Z)]$ of binary lotteries $l_p = (p, w_0, w_1)$, where Z consists of yardstick lotteries $z^k = (q_k, 0, w_1)$. Z is adapted to l_p with $|Z| = 15$. τ has a CDF-complexity representation with $u(w) = w^\alpha$ and $H(r) = (\Phi^{-1}(G(r)))^2$, for G given by (1) with $\kappa = 0, \gamma = 0.5$. Priors are distributed $Q \sim U[0, 1]$.

the presence of apparent inverse-S probability weighting in the certainty equivalents of binary lotteries as in Section 3.4, and its *absence* in uncertainty equivalent valuation tasks.

Andreoni and Sprenger (2011) document one instance of non-linearity in subjects' uncertainty equivalents: while probability equivalents are linear in p along nearly the entire unit interval, valuations of l_p exhibit a slight upward deviation from this linear relationship for p close to 1. They interpret this "certainty premium" pattern as evidence for a systematic preference for certainty (Neilson, 1992; Schmidt, 1998; Diecidue et al., 2004), in which the DM attaches a premium to riskless or near-riskless lotteries. Our model does not generate this certainty premium, consistent with the idea that individuals may have a systematic preference for certainty that operates independently of our model of tradeoff-induced noise.[9]

[9]Models capturing a preference for certainty were developed to explain the classic Allais paradox, a violation of independence in choices from binary menus that our model of CDF-complexity, which satisfies a stochastic analog of the Independence axiom in binary choice, also cannot explain.

References

Andreoni, J. and Sprenger, C. (2011). Uncertainty Equivalents: Testing the Limits of the Independence Axiom. Technical Report w17342, National Bureau of Economic Research, Cambridge, MA.

Ariely, D., Loewenstein, G., and Prelec, D. (2003). “Coherent arbitrariness”: Stable demand curves without stable preferences. *The Quarterly journal of economics*, 118(1):73–106. Publisher: MIT Press.

Bell, D. E. (1985). Disappointment in Decision Making Under Uncertainty. *Operations Research*, 33(1):1–27. Publisher: INFORMS: Institute for Operations Research.

Cohen, J., Ericson, K. M., Laibson, D., and White, J. M. (2020). Measuring Time Preferences. *Journal of Economic Literature*, 58(2):299–347.

Debreu, G. (1983). Topological methods in cardinal utility theory. In *Mathematical Economics*, pages 120–132. Cambridge University Press.

Diecidue, E., Schmidt, U., and Wakker, P. P. (2004). The Utility of Gambling Reconsidered. *Journal of Risk and Uncertainty*, 29(3):241–259.

Frederick, S. and Fischoff, B. (1998). Scope (in)sensitivity in elicited valuations. *Risk Decision and Policy*, 3(2):109–123.

Fudenberg, D., Kleinberg, J., Liang, A., and Mullainathan, S. (2022). Measuring the Completeness of Economic Models. *Journal of Political Economy*, 130(4):956–990.

Gul, F. (1991). A Theory of Disappointment Aversion. *Econometrica*, 59(3):667–686. Place: OXFORD Publisher: Econometric Society.

Gul, F. and Pesendorfer, W. (2006). Random Expected Utility. *Econometrica*, 74(1):121–146.

He, J. and Natenzon, P. (2023a). Moderate Utility. *American Economic Review: Insights*.

He, J. and Natenzon, P. (2023b). Random Choice and Differentiation.

Hsee, C. K., Blount, S., Loewenstein, G. F., and Bazerman, M. H. (2009). Preference Reversals Between Joint and Separate Evaluations of Options: A Review and Theoretical Analysis. *Psychological bulletin*, 125(5):576–590.

Kahneman, D. and Tversky, A. (1979). Prospect Theory: An Analysis of Decision Under Risk. *Econometrica (pre-1986)*, 47(2):263–291. Num Pages: 29 Place: Evanston, United Kingdom Publisher: Blackwell Publishing Ltd.

Loomes, G. and Sugden, R. (1986). Disappointment and Dynamic Consistency in Choice under Uncertainty. *The Review of Economic Studies*, 53(2):271.

Lovász, L. (1999). Hit-and-run mixes fast. *Mathematical Programming*, 86(3):443–461.

McGranaghan, C., Nielsen, K., O'Donoghue, T., Somerville, J., and Sprenger, C. D. (2024). Distinguishing Common Ratio Preferences from Common Ratio Effects Using Paired Valuation Tasks. *American Economic Review*, 114(2):307–347.

Moretti, A. C. (2003). A weighted projection centering method. *Computational & Applied Mathematics*, 22(1):19–36. Publisher: Sociedade Brasileira de Matemática Aplicada e Computacional.

Neilson, W. S. (1992). Some mixed results on boundary effects. *Economics Letters*, 39(3):275–278.

Schmidt, U. (1998). A Measurement of the Certainty Effect. *Journal of Mathematical Psychology*, 42(1):32–47.

Smith, R. L. (1984). Efficient Monte Carlo Procedures for Generating Points Uniformly Distributed over Bounded Regions. *Operations Research*, 32(6):1296. Publisher: INFORMS: Institute for Operations Research.

Toma, M. and Bell, E. (2024). Understanding and increasing policymakers' sensitivity to program impact. *Journal of Public Economics*, 234:105096.

Vallender, S. S. (1974). Calculation of the Wasserstein distance between probability distributions on the line. *Theory of Probability & Its Applications*, 18(4):784–786. Publisher: SIAM.